Non-commutative L_p spaces without the completely bounded approximation property [1]

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1. joint work with Vincent Lafforgue, CNRS

- Banach space approximation property
- Operator space approximation property
- Main results
- Overline of proof : Schur multipliers on the Schatten class

Banach space approximation property

Definition

A Banach space X has the approximation property (AP) if there exists a net

$$T_\alpha \in F(X) = \{\text{finite rank linear maps} : X \to X\}$$

such that $T_\alpha \to id_X$ uniformly on compact subsets of X.

Studied by Grothendieck, who asked : Does there exist X without AP ?

Answer : YES (Enflo 70's, artificial example).
Only natural example : $B(\ell^2)$ (Szankowski 81).

Interesting open question : Find other examples of natural Banach spaces, for which the obstruction to AP is understandable.
Conjecture : $C^*_{red}(SL(3,\mathbb{Z}))$, or $C^*(F_2)$.
Hard because it is difficult to understand bounded operators between C^*-algebras. But completely bounded operators are nicer...

Operator space approximation property

Notation : $\mathcal{K}$ is the C^*-algebra of compact operators on ℓ^2.

Definition

An operator space X has the operator space approximation property (OAP) if there is a net $T_\alpha \in F(X)$ such that $T_\alpha \otimes id \to id_{X \otimes_{min} \mathcal{K}}$ uniformly on compact subsets of $X \otimes_{min} \mathcal{K}$.

A has *CBAP* if moreover $\sup_\alpha \|T_\alpha\|_{cb} < \infty$.

Of course, CBAP $\Longrightarrow$ OAP $\Longrightarrow$ AP.

Previous results

Theorem (Haagerup '79)

If Γ *is a free group, or* $\Gamma = SL(2,\mathbb{Z})$*, then* $C^*_{red}(\Gamma)$ *has CBAP with constant* 1*.*

Theorem (Haagerup '86)

If $\Gamma = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$*, then* $C^*_{red}(\Gamma)$ *does not have CBAP. Hence, the same holds for* $SL(3,\mathbb{Z})$*.*

BUT (Haagerup and Kraus 94) $C^*_{red}(\Gamma)$ has OAP.

Main results

Main Theorem 1

$\Gamma = SL(3,\mathbb{Z})$. $C^*_{red}(\Gamma)$ does not have OAP.

Haagerup and de Laat extended Theorem 1 to Γ = lattice in a simple Lie group of higher rank and finite center.

Main Theorem 2 (real)

If $1 \leq p < 4/3$ or $4 < p < \infty$, $L_p(VN(SL(3,\mathbb{Z})))$ does not have CBAP.

Main Theorem 2 (p-adic)

Let $F = \mathbb{Q}_\ell$ (=p-adic field). If $p \neq 2$ $\exists n$ such that if Γ is a lattice in $SL(n,F)$ then $L_p(VN(\Gamma))$ does not have CBAP.

Theorem 2 $\Longrightarrow$ Theorem 1 by a result of Junge-Ruan '03 (valid with $SL(3,\mathbb{Z})$ replaced by any discrete hyperlinear group).

From know on, we study some aspects of the proof of

Main Theorem 2 (real)

If $1 \leq p < 4/3$ or $4 < p < \infty$, $L_p(VN(SL(3,\mathbb{Z})))$ does not have CBAP.

A first reduction

Let us come back to *Hard because it is difficult to understand bounded operators between C^*-algebras. But completely bounded operators are nicer...*

Proposition

Let Γ be a discrete group, and $X = C^*_{red}(\Gamma)$ or $X = L_p(VN(\Gamma))$. If X has the OAP or the CBAP, then the maps T_α can be taken as Fourier multipliers, i.e. as maps of the form m_{φ_α} for $\varphi_\alpha : \Gamma \to \mathbb{C}$ of finite support.

Fourier multiplier : $m_\varphi \lambda(s) = \varphi(s)\lambda(s)$.

Schur multipliers

For $a = (a_{i,j}) \in M_n$, Schur multiplier $\mathcal{M}_a : a \in M_n \mapsto (a_{i,j}x_{i,j}) \in M_n$.

(Bożejko-Fendler characterization) : for $\varphi : \Gamma \to \mathbb{C}$,

$$\|T_\varphi\|_{cb(C^*_\lambda(\Gamma),C^*_\lambda(\Gamma))} = \sup_{n\in\mathbb{N},s_1,\ldots,s_n,t_1,\ldots t_n\in\Gamma} \|\mathcal{M}_{(\varphi(s_i^{-1}t_j))}\|_{cb(M_n,M_n)} =: \|\varphi\|_{cb}.$$

Theorem (Haagerup, Bożejko-Fendler)

If Γ is a discrete group, $C^*_{red}(\Gamma)$ has CBAP iff Γ is weakly amenable.

A discrete locally compact group G is weakly amenable if there is a sequence $\varphi_n : G \to \mathbb{C}$ of functions with finite supportcontinuous functions with compact support such that :

- $\lim_n \varphi_n(s) = 1$ uniformly on compact subsets
- $\sup_n \|\varphi_n\|_{cb} < \infty$.

$1 < p < \infty$ case

Notation : $\|\varphi\|_{p-cb} := \sup_{n\in\mathbb{N}, s_1,\dots,s_n,t_1,\dots t_n \in \Gamma} \|\mathcal{M}_{(\varphi(s_i^{-1}t_j))}\|_{cb(S_p^n, S_p^n)}$.

Definition

A locally compact group G has the "p-variant of weak amenability" if there is a sequence $\varphi_n : G \to \mathbb{C}$ of continuous functions with compact support such that :

- $\lim_n \varphi_n(s) = 1$ uniformly on compact subsets
- $\sup_n \|\varphi_n\|_{p-cb} < \infty$.

Only one implication is known : if Γ is a **discrete** group

$$L_p(VN(\Gamma)) \text{ has CBAP} \Longrightarrow \Gamma \text{ has the "}p\text{-variant of weak amenability".}$$

A second reduction

Theorem (Haagerup for $p = \infty$,LdlS for $p < \infty$)

Let $\Gamma = SL(3, \mathbb{Z})$ *and* $G = SL(3, \mathbb{R})$*. If* $1 \leq p \leq \infty$ *then*

$$\Gamma \textit{ is } p\textit{-weakly amenable} \iff G \textit{ is } p\textit{-weakly amenable}.$$

(and this holds more generally if Γ is a lattice in a locally compact group G). Reason = induction.

We are left to prove that if $p > 4$ then $G = SL(3, \mathbb{R})$ is not "p-weakly amenable".

Working with $SL(3,\mathbb{R})$.

Consider the compact subgroup $K = SO(3,\mathbb{R}) \subset SL(3,\mathbb{R}) = G$. Remark that replacing $\varphi : G \to \mathbb{C}$ by $\widetilde{\varphi} : g \mapsto \int_{K\times K} \varphi(kgk')dkdk'$ does not increase $\|\varphi\|_{p-cb}$.

Proof : for $s_1,\dots,s_n,t_1,\dots,t_n \in G$ the matrix $(\widetilde{\varphi}(s_i^{-1}t_j))_{i,j\le n}$ is the average over $K \times K$ of the matrix $(\varphi((s_ik)^{-1}t_jk'))_{i,j\le n}$.

Hence, the following result implies that $SL(3,\mathbb{R})$ is not p-weakly amenable :

Theorem (Lafforgue-dlS)

Let $p > 4$. There is constants $C, \alpha > 0$ such that :
If $\varphi : K\backslash G/K \to \mathbb{C}$ is a continuous K-biinvariant function on G such that $\|\varphi\|_{p-cb} < \infty$, then $l = \lim_{s\to\infty} \varphi(s)$ exists, and

$$|\varphi(s) - l| \le C\|s\|^{-\alpha}\|\varphi\|_{p-cb}$$

(for $s \in M_3(\mathbb{R})$, $\|s\|$ is the operator norm).

Idea of proof of L-dlS Theorem

Main ingredient : S^2 = unit sphere in $\mathbb{R}^3$. For $-1 < \delta < 1$, consider $T_\delta : L^2(S^2) \to L^2(S^2)$ defined by

$$T_\delta f(x) = \text{ average of } f(y) \text{ on the circle } \{y, \langle x, y\rangle = \delta\}.$$

Proposition

$T_\delta \in S_p$ if and only if $p > 4$.
If $|\delta| \leq 1/2$, $\|T_\delta - T_0\|_{S_p} \leq C|\delta|^{1/2-2/p}$.

Connection with $SL(3,\mathbb{R})$: see $U = SO(2,\mathbb{R}) \subset SO(3,\mathbb{R}) = K$ by $A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$. Then $K/U \simeq S^2$ via $kU \mapsto kv$ where $v = (1,0,0) \in S^2$.
And $U\backslash K/U \simeq [-1,1]$ via $i : UkU \mapsto \langle kv, v\rangle$.
So that T_δ corresponds to $f \in L^2(K/U) \mapsto \int_U f(\cdot uk)du$ where $k \in K$ is such that $i(k) = \delta$.

Idea of proof of L-dlS Theorem (2)

(singular value decomposition) $K\backslash G/K \simeq \{\lambda_1 \geq \lambda_2 \geq \lambda_3, \sum \lambda_i = 0\}$ with the identification $\lambda_1, \lambda_2, \lambda_3 \mapsto D(\lambda) = Kdiag(e^{\lambda_1}, e^{\lambda_2}, e^{\lambda_3})K$.

For any $\lambda \in \mathbb{R}$, $k \in K \mapsto kD(\lambda, -\lambda/2, -\lambda/2)K \in G/K$ induces a map $q_\lambda : S^2 \simeq K/U \to G/K$.
Key property : For any $\lambda, \mu \in \mathbb{R}$, if $x, y \in S^2$, we have that $\langle x, y\rangle$ depends only on $q_\mu(x)^{-1}q_\lambda(y)$ in $K\backslash G/K$.

Consequence : if $\psi_{\lambda,\mu}(\delta) = \varphi(q_\mu(x)^{-1}q_\lambda(y))$ for any $x, y \in S^2$ such that $\langle x, y\rangle = \delta$, then the Schur multiplier $\mathcal{M}_{\psi_{\lambda,\mu}}$ on $S_p(L^2(S^2))$ "defined" by $(a_{x,y})_{x,y\in S^2} \mapsto (\psi_{\lambda,\mu}(\langle x, y\rangle)a_{x,y})_{x,y\in S^2}$ has norm at most $\|\varphi\|_{p-cb}$.
But $M_{\psi_{\lambda,\mu}}(T_\delta) = \psi_{\lambda,\mu}(\delta)$. So, using the previous proposition, one gets

$$|\psi_{\lambda,\mu}(\delta) - \psi_{\lambda,\mu}(0)| \leq \|\psi_{\lambda,\mu}(\delta)T_\delta - \psi_{\lambda,\mu}(0)T_0\| \leq C\sqrt{|\delta|}.$$

One concludes taking a good choice of λ, μ, δ.

Thank you for you attention !