# Regularity estimates in Hölder spaces for Schrödinger operators via a $T1$ theorem

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# The Role of $T1$ Theorem in Harmonic Analysis

It is well-known the crucial role played by $T1$ theorem in the analysis of $L^2$-boundedness of singular integrals.

The $L^2$ boundedness of convolution type singular integrals. $\leftrightarrow$ Fourier Transforms

The $L^2$ boundedness of nonconvolution type singular integrals. $\leftrightarrow$ $T1$ Theorem

# The Role of $T1$ in Harmonic Analysis

## Theorem

*For a weakly defined operator $T : C_0^\infty \to (C_0^\infty)^*$ whose kernel is a standard kernel (satisfies the classical size and smoothness conditions) and $T$ enjoys the weak boundedness property.*
*Then the following statements are equivalent:*

- *$T$ is $L^2$ bounded;*
- *$T1 \in BMO$ and $T^*1 \in BMO$.*


- G. David and J.-L. Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math. 120, 1984
- G. David; J. -L. Journé; S. Semmes. *Calderón–Zygmund operators, para-accretive functions and interpolation*, Rev. Mat. Iberoamericana 4, 1985
- T. Hytönen, *An operator-valued Tb Theorem,* J. Funct.Anal. 234, 2006
- T. Hytönen and L.Weis, *A T1 Theorem for integral transformations with operator valued kernels,* J. Reine Angew. Math. 599, 2006
- L. Grafakos, *Classical and modern Fourier analysis*, Pearson Education, Inc., Upper Saddle River, NJ, 2004.

# Why are they important?

¡¡The $L^2$-boundedness of singular integral operator
is
essentially reduced to test the behavior of $T1$ and $T^*1$!!

The Fourier transform is not needed for the $L^2$- boundedness of singular integrals.

# Classical $BMO$ Spaces

For $f \in L_{loc}(\mathbb{R}^n)$, set

$$\|f\|_{BMO} = \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| \, dx,$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^n$.

$$BMO(\mathbb{R}^n) = \{f \in L_{loc}(\mathbb{R}^n) : \|f\|_{BMO} < \infty\}.$$

$$BMO \leftrightarrow L^\infty$$

Boundedness of classical singular integral

Interpolation theory

Carleson measure

# The *BMO* boundedness related to T1 Theorem

**Theorem (Not every CZO maps $BMO_C$ into $BMO$)**

*A Calderón-Zygmund operator $T : BMO_C(\mathbb{R}^n) \to BMO(\mathbb{R}^n)$ is bounded if and only if $T1$ is a constant.*

Recently, J. J. Betancor, R. Crescimbeni, J. C. Fariña, P. R. Stinga and J. L. Torrea proved a $T1$ criterion for the $BMO_H$-boundedness of $T$.

- J. J. Betancor, R. Crescimbeni, J. C. Fariña, P. R. Stinga and J. L. Torrea, A $T1$ criterion for Hermite-Calderón-Zygmund operators on the $BMO_H(\mathbb{R}^n)$ space and applications, to appear in Ann. Sc. Norm. Sup. Pisa Cl. Sci.

# *BMO*-spaces related to Hermite operator $H$

Hermite operator: $H = -\Delta + |x|^2$.

$f \in L_{loc}(\mathbb{R}^n)$ is in $BMO_H(\mathbb{R}^n)$ if

(i) $\frac{1}{|B|}\int_B |f(x) - f_B|\,dx \le C$, for every ball $B$ in $\mathbb{R}^n$, and

(ii) $\frac{1}{|B|}\int_B |f(x)|\,dx \le C$, for every $B = B(x_0, r_0)$, where $x_0 \in \mathbb{R}^n$ and $r_0 \ge \gamma(x_0)$,

where $f_B = \frac{1}{|B|}\int_B f(x)\,dx$ and $\gamma(x) = \frac{1}{1+|x|}$.

$\|f\|_{BMO_H(\mathbb{R}^n)} = \inf\{C \ge 0 : \text{ (i) and (ii) hold}\}$.

$BMO_H(\mathbb{R}^n) = \left\{f \in L_{loc} : \|f\|_{BMO_H(\mathbb{R}^n)} < \infty\right\}$.

# Hermite-Calderón-Zygmund operator

## Definition

Let $T$ be a bounded linear operator on $L^2(\mathbb{R}^n)$ such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy, \quad f \in L^2_c(\mathbb{R}^n) \text{ and a.e. } x \notin \mathrm{supp}(f).$$

$T$ is a Hermite-Calderón-Zygmund operator if

(i) $|K(x,y)| \leq \frac{C}{|x-y|^n}\, e^{-c\left[|x||x-y|+|x-y|^2\right]}$, for all $x, y \in \mathbb{R}^n$ with $x \neq y$,

(ii) $|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \leq C \frac{|y-z|}{|x-y|^{n+1}}$, when $|x-y| > 2|y-z|$.

# A $T1$ Theorem for the Boundedness in $BMO_H$

Betancor etc. proved a $T1$ criterion for the $BMO_H$-boundedness of $T$.

**Theorem** ($T1$-type criterion for an Hermite-Calderón-Zygmund operator)

*Let $T$ be an Hermite-Calderón-Zygmund operator.*
*Then, the following two statements are equivalent:*

- *$T$ is a bounded operator on $BMO_H(\mathbb{R}^n)$;*
- *there exists $C > 0$ for which the following two conditions are satisfied*
  (i) $\frac{1}{|B(x,\gamma(x))|} \int_{B(x,\gamma(x))} |T1(y)|\, dy \leq C$, *for every* $x \in \mathbb{R}^n$, *and*
  (ii) $\left(1 + \log\left(\frac{\gamma(x)}{s}\right)\right) \frac{1}{|B(x,s)|} \int_{B(x,s)} |T1(y) - (T1)_{B(x,s)}|\, dy \leq C$, *for every* $x \in \mathbb{R}^n$ *and* $s > 0$ *such that* $0 < s \leq \gamma(x)$.

# The boundedness of the operators related with $H$

The maximal operators and Littlewood-Paley $g$-functions associated to the heat and Poisson semigroups for $H$ and the Hermite-Riesz transforms all are Hermite-Calderón-Zygmund operators. Therefore the $T1$ theorem could be applied.

In fact, as application of the above $T1$ theorem, Betancor etc. proved

**Theorem** (Harmonic analysis operators related to Hermite operator $H$)

*The maximal operators and the Littlewood-Paley $g$-functions associated with the heat $\{W_t^H\}_{t>0}$ and Poisson $\{P_t^H\}_{t>0}$ semigroups generated by $H$ and the Hermite-Riesz transforms are bounded from $BMO_H(\mathbb{R}^n)$ into itself.*

# The aims of our talk:

1. To give a $T1$-type criterion for the boundedness of operator $T$ in $BMO$-spaces related with a Schrödinger operator ;
2. To give some applications of the $T1$-type criterion with some operators in Harmonic analysis such as the maximal operators, Littlewood–Paley $g$-functions, Laplace transform type multipliers, Riesz transforms and negative powers related with a Schrödinger operator $\mathcal{L}$.

# Schrödinger operator $\mathcal{L} = -\Delta + V$

The time independent Schrödinger operator in $\mathbb{R}^n$, $n \geq 3$,

$$\mathcal{L} := -\Delta + V.$$

The nonnegative potential $V$ satisfies a reverse Hölder inequality for some $q \geq n/2$; that is, there exists a constant $C = C(q, V)$ such that

$$\left( \frac{1}{|B|} \int_B V(y)^q \, dy \right)^{1/q} \leq \frac{C}{|B|} \int_B V(y) \, dy,$$

for all balls $B \subset \mathbb{R}^n$.
Associated to this potential, Z. Shen defines the critical radii function as

$$\rho(x) := \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) \, dy \leq 1 \right\}, \qquad x \in \mathbb{R}^n.$$

We have $0 < \rho(x) < \infty$.

- Z. Shen, $L^p$ estimates for Schrödinger operators with certain potentials, Ann. Inst. Fourier (Grenoble) 45 (1995), 513–546.

# $BMO^\alpha$–spaces related to $\mathcal{L}$

$f \in L_{loc}(\mathbb{R}^n)$ is in $BMO_{\mathcal{L}}^{\alpha}$ $(0 \le \alpha \le 1)$, if

(i) $\frac{1}{|B|}\int_B |f(x) - f_B|\, dx \le C\,|B|^{\frac{\alpha}{n}}$, for every ball $B$ in $\mathbb{R}^n$, and

(ii) $\frac{1}{|B|}\int_B |f(x)|\, dx \le C\,|B|^{\frac{\alpha}{n}}$, for every $B = B(x_0, r_0)$, where $x_0 \in \mathbb{R}^n$ and $r_0 \ge \rho(x_0)$.

$\|f\|_{BMO_{\mathcal{L}}^{\alpha}} = \inf\{C > 0 : C \text{ in } (i) \text{ and } (ii)\}$.

$BMO_{\mathcal{L}}^{\alpha}(\mathbb{R}^n) = \left\{f \in L_{loc} : \|f\|_{BMO_{\mathcal{L}}^{\alpha}(\mathbb{R}^n)} < \infty\right\}$. $\quad BMO_{\mathcal{L}}^{0} = BMO_{\mathcal{L}}$.

The restriction $\alpha \le 1$ in the definition above is necessary because if $\alpha > 1$ then the space only contains constant functions.


- J. Dziubański, G. Garrigós, T. Martínez, J. L. Torrea and J. Zienkiewicz, *BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality*, Math. Z. 249 (2005), 329–356.

- B. Bongioanni, E. Harboure and O. Salinas, *Weighted inequalities for negative powers of Schrödinger operators*, J. Math. Anal. Appl. 348 (2008), 12–27.

# Campanato-type description of $BMO_{\mathcal{L}}^{\alpha}$–spaces

Let

$$C^{\alpha}(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) : |f(x) - f(y)| \le C\,|x-y|^{\alpha}\},$$

$$[f]_{C^{\alpha}} = \sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}}.$$

Recall that $BMO^{\alpha}(\mathbb{R}^n) = C^{\alpha}(\mathbb{R}^n)$ with $\|f\|_{BMO^{\alpha}(\mathbb{R}^n)} \sim [f]_{C^{\alpha}}$ and $0 < \alpha \le 1$.

- S. Campanato, Proprietà di hölderianità di alcune classi di funzioni, Ann. Scuola Norm. Sup. Pisa (3) 17 (1963), 175–188.

In our case there is s similar identification.

**Proposition (Campanato-type description of $BMO_{\mathcal{L}}^{\alpha}$)**

*Let $0 < \alpha \le 1$. A function $f$ belongs to $BMO_{\mathcal{L}}^{\alpha}$ if and only if $f \in C^{\alpha}(\mathbb{R}^n)$ and $|f(x)| \le C\rho(x)^{\alpha}$, for all $x \in \mathbb{R}^n$. Moreover, $\|f\|_{BMO_{\mathcal{L}}^{\alpha}} \sim [f]_{C^{\alpha}(\mathbb{R}^n)} + \|f\rho^{-\alpha}\|_{L^{\infty}(\mathbb{R}^n)}$.*

- B. Bongioanni, E. Harboure and O. Salinas, Weighted inequalities for negative powers of Schrödinger operators, J. Math. Anal. Appl. 348 (2008), 12–27.

# $\gamma$-Schrödinger-Calderón-Zygmund operators

## Definition ($\gamma$-Schrödinger-Calderón-Zygmund operators)

*Let* $0 \le \gamma < n$, $1 < p \le q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$. *Let* $T$ *be a bounded linear operator from* $L^p(\mathbb{R}^n)$ *into* $L^q(\mathbb{R}^n)$ *such that*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy, \quad f \in L^p_c(\mathbb{R}^n) \text{ and a.e. } x \notin supp(f).$$

$T$ *is a* $\gamma$*-Schrödinger-Calderón-Zygmund operator with smoothness exponent* $\delta > 0$ *if for some constant* $C$

(1) $|K(x,y)| \le \dfrac{C}{|x-y|^{n-\gamma}} \left(1 + \dfrac{|x-y|}{\rho(x)}\right)^{-N}$, *for all* $N > 0$ *and* $x \ne y$, *(size)*

(2) $|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \le C \dfrac{|y-z|^{\delta}}{|x-y|^{n-\gamma+\delta}}$, *when* $|x-y| > 2|y-z|$. *(smoothness)*

Note that every Hermite-Calderón-Zygmund operator is a classical Calderón-Zygmund operator.

# Definition of $T1$

Suppose that $f \in BMO_{\mathcal{L}}^{\alpha}$ and $R \geq \rho(x_0)$, $x_0 \in \mathbb{R}^n$. We define

$$Tf(x) = T\left(f\chi_{B(x_0,R)}\right)(x) + \int_{B(x_0,R)^c} K(x,y)f(y)\,dy, \quad \text{a.e. } x \in B(x_0,R).$$

The first term in the right hand side makes sense since $f\chi_{B(x_0,R)} \in L_c^p(\mathbb{R}^n)$.

The second term is absolutely convergent by the size condition of the kernel which is different from the case of classical Laplacian.

So, $T1$ is well defined.

# Main results: two $T1$-type criterions

**Theorem** ($T1$-type criterion in $BMO_{\mathcal{L}}^{\alpha}$, $0 < \alpha < 1$)

*Let $T$ be a $\gamma$-Schrödinger-Calderón-Zygmund operator, $\gamma \geq 0$, with smoothness exponent $\delta$, such that $\alpha + \gamma < \min\{1, \delta\}$. Then, the following statements are equivalent:*

- *$T$ is bounded from $BMO_{\mathcal{L}}^{\alpha}$ into $BMO_{\mathcal{L}}^{\alpha+\gamma}$;*
- *there exists a constant $C$ such that*

$$\left(\frac{\rho(x)}{s}\right)^{\alpha} \frac{1}{|B|^{1+\frac{\gamma}{n}}} \int_B |T1(y) - (T1)_B| \, dy \leq C,$$

*for every ball $B = B(x, s)$, $x \in \mathbb{R}^n$ and $0 < s \leq \frac{1}{2}\rho(x)$.*

# $T1$ Theorem of Betancor etc.'s

**Theorem** ($T1$-type criterion for an Hermite-Calderón-Zygmund operator)

*Let $T$ be an Hermite-Calderón-Zygmund operator.*
*Then, the following two statements are equivalent:*

- *$T$ is a bounded operator on $BMO_H(\mathbb{R}^n)$;*
- *there exists $C > 0$ for which the following two conditions are satisfied*
  (i) $\frac{1}{|B(x,\gamma(x))|}\int_{B(x,\gamma(x))} |T1(y)|\, dy \le C$, *for every* $x \in \mathbb{R}^n$, *and*
  (ii) $\left(1+\log\left(\frac{\gamma(x)}{s}\right)\right)\frac{1}{|B(x,s)|}\int_{B(x,s)} |T1(y)-(T1)_{B(x,s)}|\, dy \le C$, *for every* $x \in \mathbb{R}^n$ *and* $s > 0$ *such that* $0 < s \le \gamma(x)$.

# Main results: two $T1$-type criterions

## Theorem ($T1$-type criterion in $BMO_{\mathcal{L}}$, $(\alpha = 0)$)

*Let $T$ be a $\gamma$-Schrödinger-Calderón-Zygmund operator, $0 \le \gamma < \min\{1, \delta\}$, with smoothness exponent $\delta$. Then, the following statements are equivalent:*

- *$T$ is a bounded operator from $BMO_{\mathcal{L}}$ into $BMO_{\mathcal{L}}^{\gamma}$;*
- *there exists a constant $C$ such that*

$$\log\left(\frac{\rho(x)}{s}\right)\frac{1}{|B|^{1+\frac{\gamma}{n}}}\int_B |T1(y) - (T1)_B|\, dy \le C,$$

*for every ball $B = B(x, s)$, $x \in \mathbb{R}^n$ and $0 < s \le \frac{1}{2}\rho(x)$.*

For any $0 < \alpha \le 1$, if $0 < s \le \frac{1}{2}\rho(x)$ then

$$1 + \log\frac{\rho(x)}{s} \sim \log\frac{\rho(x)}{s} \quad \text{and} \quad 1 + \frac{2^{\alpha}\left(\left(\frac{\rho(x)}{s}\right)^{\alpha} - 1\right)\log 2}{2^{\alpha}-1} \sim \left(\frac{\rho(x)}{s}\right)^{\alpha}.$$

And $\lim_{\alpha\to 0} 1 + \frac{2^{\alpha}\left(\left(\frac{\rho(x)}{s}\right)^{\alpha} - 1\right)\log 2}{2^{\alpha}-1} = 1 + \log\frac{\rho(x)}{s}$.

Therefore, the criterion of the case $(\alpha = 0)$ is indeed the limit case of the criterion of the case $(0 < \alpha < 1)$.

# Main results: two $T1$-type criterions

We should note that our results are more general than the results of Betancor,Crescimbeni, Farina. Stinga and Torrea.

1. Assumption: $T : L^p \to L^q (1 < p \leq q < \infty)$ instead of $T : L^2 \to L^2$;
2. Result: $T : BMO_{\mathcal{L}}^{\alpha} \to BMO_{\mathcal{L}}^{\alpha+\gamma}$ instead of $T : BMO_H \to BMO_H$.

# Application: Pointwise multipliers in $BMO_{\mathcal{L}}^{\alpha}, 0 \leq \alpha < 1$

## Proposition (Pointwise Multipliers)

*Let $\psi$ be a measurable function on $\mathbb{R}^n$. We denote by $T_\psi$ the multiplier operator defined by $T_\psi(f) = f\psi$. Then*

(A) *$T_\psi$ is a bounded operator in $BMO_{\mathcal{L}}$ if and only if $\psi \in L^\infty(\mathbb{R}^n)$ and there exists $C > 0$ such that, for all balls $B = B(x_0, s)$ with $0 < s < \frac{1}{2}\rho(x_0)$,*

$$\log\left(\frac{\rho(x_0)}{s}\right)\frac{1}{|B|}\int_B |\psi(y) - \psi_B|\, dy \leq C.$$

(B) *$T_\psi$ is a bounded operator in $BMO_{\mathcal{L}}^{\alpha}$, $0 < \alpha < 1$, if and only if $\psi \in L^\infty(\mathbb{R}^n)$ and there exists $C > 0$ such that, for all balls $B = B(x_0, s)$ with $0 < s < \frac{1}{2}\rho(x_0)$,*

$$\left(\frac{\rho(x_0)}{s}\right)^{\alpha}\frac{1}{|B|}\int_B |\psi(y) - \psi_B|\, dy \leq C.$$

**Semigroups related with $\mathcal{L}$**

- The heat–diffusion semigroup $\mathcal{W}_t \equiv e^{-t\mathcal{L}}$: $\mathcal{W}_t$ is the solution of heat equation:

$$\partial_t u + \mathcal{L}u = 0.$$

$$\mathcal{W}_t f(x) \equiv e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^n} \mathcal{W}_t(x,y) f(y)\, dy, \qquad f \in L^2(\mathbb{R}^n),\ x \in \mathbb{R}^n,\ t > 0.$$

- The generalized Poisson semigroups $\mathcal{P}_t^\sigma$:

$$\mathcal{P}_t^\sigma f(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-r} \mathcal{W}_{\frac{t^2}{4r}} f(x)\, \frac{dr}{r^{1-\sigma}}.$$

When $\sigma = \frac{1}{2}$, $\mathcal{P}_t^\sigma$ is the classical Poisson operator $\mathcal{P}_t$ which is a solution of the equation: $\partial_{tt} u = \mathcal{L}u.$

And $\mathcal{P}_t^\sigma$ is a solution of the equation:

$$-\mathcal{L}_x u + \frac{1-2\sigma}{t} \partial_t u + \partial_{tt} u = 0.$$

It is related with the extension problem for the fractional Laplacian.

**Maximal operators for the heat–diffusion semigroup $\mathcal{W}_t \equiv e^{-t\mathcal{L}}$:**

$$\mathcal{W}^* f(x) = \sup_{t>0} |\mathcal{W}_t f(x)| = \|\mathcal{W}_t f\|_{L^\infty((0,\infty),dt)} .$$

**Maximal operators for the generalized Poisson operators $\mathcal{P}_t^\sigma$:**

$$\mathcal{P}_t^{\sigma,*} f(x) = \sup_{t>0} |\mathcal{P}_t^\sigma f(x)| = \|\mathcal{P}_t^\sigma f(x)\|_{L^\infty((0,\infty),dt)} .$$

**Littlewood–Paley $g$-function for the heat–diffusion semigroup:**

$$g_{\mathcal{W}}(f)(x) = \left(\int_0^\infty |t\partial_t \mathcal{W}_t f(x)|^2 \frac{dt}{t}\right)^{1/2} = \|t\partial_t \mathcal{W}_t f(x)\|_{L^2\left((0,\infty),\frac{dt}{t}\right)} .$$

**Littlewood–Paley $g$-function for the Poisson semigroup:**

$$g_{\mathcal{P}}(f)(x) = \left(\int_0^\infty |t\partial_t \mathcal{P}_t f(x)|^2 \frac{dt}{t}\right)^{1/2} = \|t\partial_t \mathcal{P}_t f(x)\|_{L^2\left((0,\infty),\frac{dt}{t}\right)} .$$

**Laplace transform type multipliers:**

$$m(\mathcal{L})f(x) = \int_0^\infty a(t)\mathcal{L}e^{-t\mathcal{L}} f(x)\, dt = \int_0^\infty a(t)\partial_t \mathcal{W}_t f(x)\, dt,$$

where $a$ is a bounded function on $[0,\infty)$ and $m(\lambda) = \lambda \int_0^\infty a(t)e^{-t\lambda}\, dt$.

# Applications:
## The regularity estimates of some operators

We can get the regularity estimates of the above operators by proving that they are $\gamma$-Schrödinger-Calderón-Zygmund operators and satisfy the conditions of the $T1$-type criterions.

**Theorem (Regularity Estimates)**

*Let $0 \leq \alpha < \min\{1, 2 - \frac{n}{q}\}$. The maximal operators associated with the heat semigroup $\{\mathcal{W}_t\}_{t>0}$ and with the generalized Poisson operators $\{\mathcal{P}_t^\sigma\}_{t>0}$, the Littlewood-Paley g-functions given in terms of the heat and the Poisson semigroups, and the Laplace transform type multipliers $m(\mathcal{L})$, are bounded from $BMO_{\mathcal{L}}^\alpha$ into itself.*

The $T1$ Theorem can also be applied to Riesz Transforms related to $\mathcal{L}$ and Negative Powers $\mathcal{L}^{-\gamma}$, $\gamma > 0$.

# Sketch proof of $T1$-type criterion for $BMO_{\mathcal{L}}^{\alpha}$, $0 < \alpha < 1$

Proof:
First we shall see that the condition on $T1$ implies that $T$ is bounded from $BMO_{\mathcal{L}}^{\alpha}$ into $BMO_{\mathcal{L}}^{\alpha+\gamma}$.In order to do this, we will show that there exists $C > 0$ such that the properties $(A_k)$ and $(B_k)$ stated in the lemma of Boundedness criterion hold for every $k \in \mathbb{N}$ and $f \in BMO_{\mathcal{L}}^{\alpha}$.

## Lemma (Boundedness criterion)

*Let $S$ be a linear operator defined on $BMO_{\mathcal{L}}^{\alpha}$, $0 \le \alpha \le 1$. Then $S$ is bounded from $BMO_{\mathcal{L}}^{\alpha}$ into $BMO_{\mathcal{L}}^{\alpha+\gamma}$, $\alpha + \gamma \le 1$, $\gamma \ge 0$, if there exists $C > 0$ such that, for every $f \in BMO_{\mathcal{L}}^{\alpha}$ and $k \in \mathbb{N}$,*

$(A_k)$ $\dfrac{1}{|Q_k|^{1+\frac{\alpha+\gamma}{n}}} \displaystyle\int_{Q_k} |Sf(x)|\, dx \le C\|f\|_{BMO_{\mathcal{L}}^{\alpha}}$, *and*

$(B_k)$ $\|Sf\|_{BMO^{\alpha+\gamma}(Q_k^*)} \le C\|f\|_{BMO_{\mathcal{L}}^{\alpha}}$, *where $BMO^{\alpha}(Q_k^*)$ denotes the usual $BMO^{\alpha}$ space on the ball $Q_k^*$.*

# Sketch proof of $T1$-type criterion for $BMO_{\mathcal{L}}^{\alpha}$, $0 < \alpha < 1$

We begin with $(A_k)$. We can divide $Tf$ as

$$Tf(x) = T\left((f - f_{Q_k})\chi_{Q_k^{***}}\right)(x) + \int_{(Q_k^{***})^c} K(x,y)(f(y) - f_{Q_k})\, dy + f_{Q_k} T1(x),$$

a.e. $x \in Q_k$.

As $T$ maps $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$, by Hölder's inequality, we have

$$\frac{1}{|Q_k|^{1+\frac{\alpha+\gamma}{n}}} \int_{Q_k} \left| T\left((f - f_{Q_k})\chi_{Q_k^{***}}\right)(x)\right| dx \le C\|f\|_{BMO_{\mathcal{L}}^{\alpha}}.$$

# Sketch proof of $T1$-type criterion for $BMO_{\mathcal{L}}^{\alpha}$, $0 < \alpha < 1$

On the other hand, given $x \in Q_k$, we have $\rho(x) \sim \rho(x_k)$ and if $|x_k - y| > 2^j \rho(x_k)$, $j \in \mathbb{N}$, then $|x - y| \geq 2^{j-1}\rho(x_k)$. By the size condition of the kernel $K$, for any $N > \alpha$ we also have

$$\frac{1}{|Q_k|^{\frac{\alpha+\gamma}{n}}} \left| \int_{(Q_k^{***})^c} K(x,y)\big(f(y) - f_{Q_k}\big)\, dy \right| \leq C\|f\|_{BMO_{\mathcal{L}}^{\alpha}}.$$

Finally,

$$\frac{1}{|Q_k|^{1+\frac{\alpha+\gamma}{n}}} \int_{Q_k} |f_{Q_k} T1(x)|\, dx = \frac{|f_{Q_k}|}{|Q_k|^{\frac{\alpha}{n}}} \frac{1}{|Q_k|^{1+\frac{\gamma}{n}}} \int_{Q_k} |T1(x)|\, dx \leq C\|f\|_{BMO_{\mathcal{L}}^{\alpha}}.$$

Hence, we conclude that $(A_k)$ holds for $T$ with a constant $C$ that does not depend on $k$.
For $(B_k)$, we can deal with it similarly but by using the $T1$-condition in the third part.

# Sketch proof of $T1$-type criterion for $BMO_{\mathcal{L}}^{\alpha}$, $0<\alpha<1$

Let us now prove the converse statement. We need a lemma which provides examples of functions that are uniformly bounded in $BMO_{\mathcal{L}}^{\alpha}$.

## Lemma (Some examples)

*There exists constants $C, C_\alpha > 0$ such that for every $x_0 \in \mathbb{R}^n$ and $0 < s \le \rho(x_0)$,*

(a) *the function*

$$g_{x_0,s}(x) := \chi_{[0,s]}(|x-x_0|)\log\left(\frac{\rho(x_0)}{s}\right) + \chi_{(s,\rho(x_0))]}(|x-x_0|)\log\left(\frac{\rho(x_0)}{|x-x_0|}\right),$$

*$x \in \mathbb{R}^n$, belongs to $BMO_{\mathcal{L}}$ and $\|g_{x_0,s}\|_{BMO_{\mathcal{L}}} \le C$;*

(b) *the function*

$$f_{x_0,s}(x) = \chi_{[0,s]}(|x-x_0|)\left(\rho(x_0)^\alpha - s^\alpha\right) + \chi_{(s,\rho(x_0))]}(|x-x_0|)\left(\rho(x_0)^\alpha - |x-x_0|^\alpha\right),$$

*$x \in \mathbb{R}^n$, belongs to $BMO_{\mathcal{L}}^{\alpha}$, $0 < \alpha \le 1$, and $\|f_{x_0,s}\|_{BMO_{\mathcal{L}}^{\alpha}} \le C_\alpha$.*

# Sketch proof of $T1$-type criterion for $BMO^{\alpha}_{\mathcal{L}}$, $0 < \alpha < 1$

Suppose that $T$ is bounded from $BMO^{\alpha}_{\mathcal{L}}$ into $BMO^{\alpha+\gamma}_{\mathcal{L}}$. Let $x_0 \in \mathbb{R}^n$ and $0 < s \leq \frac{1}{2}\rho(x_0)$ and $B = B(x_0, s)$. For such $x_0$ and $s$ consider the nonnegative function $f_0(x) \equiv f_{x_0,s}(x)$ defined in the lemma above. Using the decomposition

$$f_0 = (f_0 - (f_0)_B)\chi_{B^{***}} + (f_0 - (f_0)_B)\chi_{(B^{***})^c} + (f_0)_B =: f_1 + f_2 + (f_0)_B$$

we can write $(f_0)_B T1(y) = Tf_0(y) - Tf_1(y) - Tf_2(y)$. So, we can get the estimation of

$$(f_0)_B \frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}} \int_B |T1(y) - (T1)_B| \, dy \leq \sum_{i=0}^{2} \frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}} \int_B |Tf_i(y) - (Tf_i)_B| \, dy$$

as in $(A_k)$. We complete the proof.
The proof of the case $\alpha = 0$ is almost the same as the proof of the theorem of $0 < \alpha < 1$ by using the example function $g_{x_0,s}(x)$.

# Acknowledgements

Thanks for your attention!