On *K*-theory of some Noncommutative Orbifold(joint work with Xiang Tang)

Yi-Jun Yao

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Operator Spaces, Quantum Probability and Applications Wuhan, June 5th, 2012

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3 Applications

- Noncommutative toroidal orbifolds
- θ deformation

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Strict Deformation Our work

Herman Weyl(1885 - 1955)



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Weyl product

$$\begin{array}{ll} f,g\in \mathcal{S}(\mathbb{R}^2),\\ (f*^Wg)(x,y) &=& \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f(x+u_1,y+u_2)g(x+v_1,y+v_2)\\ & \mathrm{e}^{2\pi\mathrm{i}(u_1v_2-u_2v_1)}\,\mathrm{d}^2u\mathrm{d}^2v. \end{array}$$

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Associative noncommutative product.

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Strict Deformation Our work

Marc Rieffel(1937 -)



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Rieffel :

• C^* -algebra $A, \alpha : \mathbb{R}^2 \to Aut(A);$

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$$\boldsymbol{a} \ast \boldsymbol{b} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \alpha_u(\boldsymbol{a}) \alpha_v(\boldsymbol{b}) \mathrm{e}^{2\pi \mathrm{i}(u_1 v_2 - u_2 v_1)} \mathrm{d}^2 u \mathrm{d}^2 v.$$

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• we complete it into a C^* -algebra \rightsquigarrow strict deformation.

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Rieffel :

- C^* -algebra $A, \alpha : \mathbb{R}^2 \to Aut(A);$
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- we complete it into a C^* -algebra \rightsquigarrow strict deformation.
- *K*-theory of the deformed algebra is the same as the original one.

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Noncommutative 2-torus

$$A_{\theta} = C(\mathbb{T}^2_{\theta}) = \langle U, V | VU = \exp 2\pi i \theta UV \rangle.$$

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Noncommutative 2-torus

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Example

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Noncommutative 2-torus

$$egin{aligned} \mathsf{A}_{ heta} = \mathsf{C}(\mathbb{T}^2_{ heta}) = \langle \mathsf{U}, \mathsf{V} | \; \mathsf{V} \mathsf{U} = \exp 2\pi \mathrm{i} heta \mathsf{U} \mathsf{V}
angle. \end{aligned}$$

Example

•
$$\theta = 0, A_0 = C(\mathbb{T}^2);$$

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Noncommutative 2-torus

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Example

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$$\theta = 0, A_0 = C(\mathbb{T}^2);$$

• $\theta \in \mathbb{Q}, A_\theta \stackrel{\text{s.Morita}}{\cong} C(\mathbb{T}^2)$

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Noncommutative 2-torus

$$A_{ heta} = C(\mathbb{T}^2_{ heta}) = \langle U, V | VU = \exp 2\pi \mathrm{i}\theta UV \rangle.$$

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$$\theta = 0, A_0 = C(\mathbb{T}^2);$$

•
$$\theta \in \mathbb{Q}, A_{\theta} \stackrel{\text{s.Monta}}{\cong} C(\mathbb{T}^2)$$

• $\theta \notin \mathbb{Q}$, $A_{\theta} = C(S^1) \rtimes_{\theta} \mathbb{Z}$, irrational rotation algebra.

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The question

Assume on the C^* -algebra A there is a strongly continuous action α of \mathbb{R}^n , plus a strongly continuous action β of a compact group G, then what would be the K-theory of the ("deformed") algebra ?

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When the two actions commute...

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When the two actions commute...

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- $\alpha \xrightarrow{\text{lift}}$ strongly continuous action $\tilde{\alpha}$ on $A \rtimes_{\beta} G$.

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- If the **G**-action β commutes with the \mathbb{R}^n -action α ,
- $\alpha \xrightarrow{\text{lift}}$ strongly continuous action $\tilde{\alpha}$ on $A \rtimes_{\beta} G$.
- Applying Rieffel's consctruction for the ℝⁿ-action α̃ on A ⋊_β G → quantized algebra (A ⋊_β G)_J.

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- $\alpha \xrightarrow{\text{lift}}$ strongly continuous action $\tilde{\alpha}$ on $A \rtimes_{\beta} G$.
- Applying Rieffel's consctruction for the ℝⁿ-action α̃ on *A* ⋊_β *G* → quantized algebra (*A* ⋊_β *G*)_{*J*}.
- " $[\alpha, \beta] = 0$ " $\Rightarrow \beta \xrightarrow{\text{lift}}$ strongly continuous action $\tilde{\beta}$ on A_J , $A_J \rtimes_{\tilde{\beta}} \mathbf{G} \simeq (\mathbf{A} \rtimes_{\beta} \mathbf{G})_J.$

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- Applying Rieffel's consctruction for the ℝⁿ-action α̃ on A ⋊_β G → quantized algebra (A ⋊_β G)_J.
- " $[\alpha, \beta] = 0$ " $\Rightarrow \beta \xrightarrow{\text{lift}}$ strongly continuous action $\tilde{\beta}$ on A_J , $A_J \rtimes_{\tilde{\beta}} G \simeq (A \rtimes_{\beta} G)_J.$
- $K_{\bullet}(A \rtimes_{\beta} G) = K_{\bullet}((A \rtimes_{\beta} G)_J) = K_{\bullet}(A_J \rtimes_{\tilde{\beta}} G).$

The general case I

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The general case I

G = Z₂ = Z/2Z−action on Rⁿ: reflection with respect to a hyperplane.

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- G = Z₂ = Z/2Z−action on Rⁿ: reflection with respect to a hyperplane.
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- $A = C(\mathbb{T}^{2n})$, *J* the standard symplectic matrix on \mathbb{R}^{2n} .

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- α (resp. β): action of \mathbb{R}^{2n} (resp. \mathbb{Z}_2) on A, dual to its action on \mathbb{T}^{2n} .

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- α (resp. β): action of \mathbb{R}^{2n} (resp. \mathbb{Z}_2) on A, dual to its action on \mathbb{T}^{2n} .
- ρ : natural inclusion $\mathbb{Z}_2 \hookrightarrow SL_{2n}(\mathbb{R}, J)$, we have

$$\beta_{g}\alpha_{\mathbf{x}} = \alpha_{\rho_{g}(\mathbf{x})}\beta_{g}, \text{ for all } \mathbf{g} \in \mathbf{G}, \mathbf{x} \in \mathbb{R}^{n}.$$

The general case II

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The general case II

"[α, β] ≠ 0" + nontrivial ρ : G → SL_n(ℝ, J), the ℝⁿ-action α on A cannot be lifted naturally to an action on A ×_β G.

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The general case II

- "[α, β] ≠ 0" + nontrivial ρ : G → SL_n(ℝ, J), the ℝⁿ-action α on A cannot be lifted naturally to an action on A ×_β G.
- we still have

$$\beta_g(a \times_J b) = \beta_g(a) \times_J \beta_g(b), \ \beta_g(a^*) = \beta_g(a)^*,$$

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- i.e., the G-action β on A_J is still well-defined.
- Therefore we can consider the crossed product algebra $A_J \rtimes_{\beta} G$.

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Main Result

Theorem (X.Tang-Y.)

When A is a separable C*-algebra, and if the actions α , β and the group homomorphism ρ satisfy

$$eta_{g}lpha_{\mathbf{x}}=lpha_{
hog}(\mathbf{x})eta_{g}, \hspace{1em} ext{for any } g\in \mathbf{G}, \mathbf{x}\in \mathbb{R}^{n}.$$

Then

$$K_{\bullet}(A_J \rtimes_{\beta} G) \cong K_{\bullet}(A \rtimes_{\beta} G), \quad \bullet = 0, 1.$$

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First part of the proof I

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First part of the proof I

B^A = {f : ℝⁿ → A smooth, f and all its derivatives being bounded }.

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- B^A = {f : ℝⁿ → A smooth, f and all its derivatives being bounded }.
- S^A : space of all *A*-valued Schwartz functions on \mathbb{R}^n .

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- B^A = {f : ℝⁿ → A smooth, f and all its derivatives being bounded }.
- S^A : space of all *A*-valued Schwartz functions on \mathbb{R}^n .
- $\langle f, g \rangle_A := \int f(x)^* g(x) dx \rightsquigarrow A$ -valued inner product on \mathcal{S}^A .

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- For given J, we define on \mathcal{B}^A

$$(F imes_J G)(x) := \int F(x + Ju) G(x + v) e^{2\pi i u \cdot v} du dv, \qquad F, G \in \mathcal{B}^A.$$

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- For given J, we define on \mathcal{B}^A

$$(F imes_J G)(x) := \int F(x + Ju) G(x + v) e^{2\pi i u \cdot v} du dv, \qquad F, G \in \mathcal{B}^{\mathcal{A}}.$$

• \mathcal{B}^A also acts on \mathcal{S}^A : $(L_F^J f)(x) := \int F(x+Ju)f(x+v)e^{2\pi i u \cdot v} du dv, \qquad F \in \mathcal{B}^A, f \in \mathcal{S}^A.$

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- A-valued inner product \sim operator norm $|| ||_J$ on \mathcal{B}^A

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- A-valued inner product \rightsquigarrow operator norm $|| ||_J$ on $\mathcal{B}^A \rightsquigarrow (\mathcal{B}^A_J, \times_J, || ||_J)$ a pre- C^* -algebra.

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- A-valued inner product \rightsquigarrow operator norm $\| \|_J$ on $\mathcal{B}^A \rightsquigarrow$ $(\mathcal{B}_J^A, \times_J, \| \|_J)$ a pre- C^* -algebra. \rightsquigarrow corresponding C^* -algebra $\overline{\mathcal{B}}_J^A$, completion of $\mathcal{S}_J^A = \overline{\mathcal{S}}_J^A$

First part of the proof II

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First part of the proof II

• \mathbb{R}^n -action α on A



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First part of the proof II

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First part of the proof II

• \mathbb{R}^n -action α on $A \rightsquigarrow$ strongly continuous \mathbb{R}^n -action ν on $\overline{\mathcal{B}}_J^A(\mathbb{R}^n$ also acts on $\overline{\mathcal{S}}_J^A$)

 $(\nu_t(F))(\mathbf{x}) := \alpha_t(F(\mathbf{x} - t)).$

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First part of the proof II

$$(\nu_t(F))(\mathbf{x}) := \alpha_t(F(\mathbf{x}-t)).$$

the fixed point subalgebra of ν can be identified with the subalgebra of B
 ^A generated by elements of the form

$$\tilde{a}(\mathbf{x}) := \alpha_{\mathbf{x}}(\mathbf{a}), \qquad \mathbf{a} \in \mathbf{A}^{\infty}.$$

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• It is exactly A_J .

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First part of the proof III

• A_J and $\overline{\mathcal{S}}_J^A \rtimes_{\nu} \mathbb{R}^n$ are strongly Morita equivalent.(Rieffel)

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First part of the proof III

- A_J and $\overline{\mathcal{S}}_J^A \rtimes_{\nu} \mathbb{R}^n$ are strongly Morita equivalent.(Rieffel)
- We can generalize it to the equivariant case(for the *G*-action β).

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- **G**-action $\overline{\beta}$ is strongly continuous.

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Proposition

the crossed product algebras $A_J \rtimes_{\beta} G$ and $(\overline{S}_J^A \rtimes_{\nu} \mathbb{R}^n) \rtimes_{\overline{\beta}} G$ are strongly Morita equivalent.

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the crossed product algebras $A_J \rtimes_{\beta} G$ and $(\overline{S}_J^A \rtimes_{\nu} \mathbb{R}^n) \rtimes_{\overline{\beta}} G$ are strongly Morita equivalent.

• By Morita equivalence, we have

$$\mathcal{K}_{ullet}(\mathcal{A}_J \rtimes_eta \mathcal{G}) \cong \mathcal{K}_{ullet}((\overline{\mathcal{S}}_J^\mathcal{A} \rtimes_
u \mathbb{R}^n) \rtimes_{\overline{eta}} \mathcal{G}).$$

Second part of the proof

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Second part of the proof

• \mathbb{C}_n : the complexified Clifford algebra associated to \mathbb{R}^n .

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Theorem

Assume \mathbb{R}^n and G act strongly continuously on the C*-algebra A, denoted by α and β , respectively. Let $\rho : G \to GL(n, \mathbb{R})$. If for any $g \in G, x \in \mathbb{R}^n$, α and β satisfy $\beta_g \alpha_x = \alpha_{\rho_g(x)} \beta_g$, then

$$\begin{array}{rcl} \mathsf{K}_{\bullet}(((\mathsf{A}\otimes\mathbb{C}_n)\rtimes_{\alpha}\mathbb{R}^n)\rtimes_{\beta}\mathsf{G}) &\cong& \mathsf{K}_{\bullet}^{\mathsf{G}}\big((\mathsf{A}\otimes\mathbb{C}_n)\rtimes_{\alpha}\mathbb{R}^n\big)\cong\mathsf{K}_{\bullet}^{\mathsf{G}}(\mathsf{A})\\ &\cong& \mathsf{K}_{\bullet}(\mathsf{A}\rtimes_{\beta}\mathsf{G}), \end{array}$$

where \mathbb{C}_n is the complexified Clifford algebra associated to \mathbb{R}^n .

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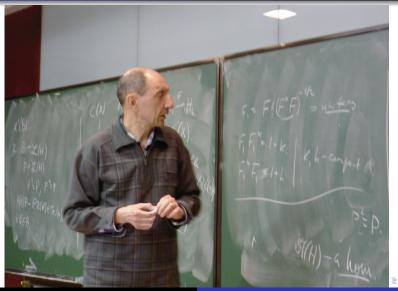
• We obtain: $\mathcal{K}_{\bullet}((\overline{\mathcal{S}}_{J}^{A} \otimes \mathbb{C}_{n}) \rtimes_{\beta} G) \cong \mathcal{K}_{\bullet+n}((\overline{\mathcal{S}}_{J}^{A} \rtimes_{\nu} \mathbb{R}^{n}) \rtimes_{\overline{\beta}} G).$

Gennadi Kasparov(1948 -)



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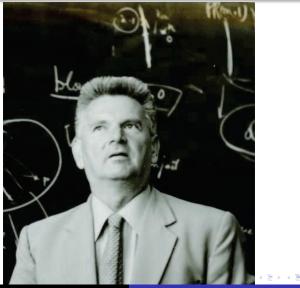


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Third part of the proof I

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Third part of the proof I

K: algebra of compact operators on separable Hilbert space; *V*₀: kernel of *J* in ℝⁿ.

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- We can suppose *G* preserves the standard complex structure on *U*.
- (Rieffel)For A = C, S
 ^C
 _J = space of compact operators on the subspace H of L²(U) generated by the elements of the form

$$g(\bar{z})e^{-\frac{\|z\|^2}{2}},$$

where g is an anti-holomorphic function.

Third part of the proof II

• $(A \otimes K \otimes C_{\infty}(V_0) \otimes \mathbb{C}_n) \rtimes_{\bar{\beta}} G \stackrel{\text{s.Morita}}{\cong} (A \otimes C_{\infty}(V_0) \otimes \mathbb{C}_n) \rtimes_{\bar{\beta}} G.$ (*G*-equivariant Morita equivalence, Combes)

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- (A ⊗ C_∞(V₀) ⊗ C_n) ⋊_{β̄} G and (A ⊗ C_∞(V₀) ⊗ C_{V₀}) ⋊_{β̄} G have the same KK-theory.

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- (A ⊗ C_∞(V₀) ⊗ C_n) ⋊_{β̄} G and (A ⊗ C_∞(V₀) ⊗ C_{V₀}) ⋊_{β̄} G have the same KK-theory.
- *G*-equivriant Thom isomorphism \Rightarrow

$$\begin{split} \mathsf{K}_{\bullet}((\overline{\mathcal{S}}_{J}^{\mathsf{A}}\otimes\mathbb{C}_{n})\rtimes_{\bar{\beta}}G) &= \mathsf{K}_{\bullet}((\mathsf{A}\otimes\mathcal{K}\otimes\mathsf{C}_{\infty}(\mathsf{V}_{0})\otimes\mathbb{C}_{n})\rtimes_{\bar{\beta}}G) \\ &= \mathsf{K}_{\bullet}((\mathsf{A}\otimes\mathsf{C}_{\infty}(\mathsf{V}_{0})\otimes\mathbb{C}_{\mathsf{V}_{0}})\rtimes_{\bar{\beta}}G) \\ &= \mathsf{K}_{\bullet}(\mathsf{A}\rtimes_{\beta}G). \end{split}$$

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Conclusion

$$\mathcal{K}_{\bullet}(\mathcal{A}_{J} \rtimes_{\beta} \mathcal{G}) \quad \stackrel{\text{1st step}}{==} \quad \mathcal{K}_{\bullet}((\overline{\mathcal{S}}_{J}^{\mathcal{A}} \rtimes_{\nu} \mathbb{R}^{n}) \rtimes_{\bar{\beta}} \mathcal{G})$$

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Conclusion

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Noncommutative toroidal orbifolds θ deformation

• 2-torus $\mathbb{T}^2 \simeq \mathbb{R}^2 / \mathbb{Z}^2 \rightsquigarrow$ an action α of \mathbb{R}^2 by translation.

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Noncommutative toroidal orbifolds θ deformation

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$$\sigma_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$
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- Z_i = the cyclic subgroups of SL(2, Z) generated by σ_i, with corresponding indices i = 2, 3, 4, 6.
- In this example $SL(2, J) = SL(2, \mathbb{R})$.
- We can define the inclusion $\rho : \mathbb{Z}_i \to SL(2, \mathbb{R})$.

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It is easy to verify that the \mathbb{Z}_i -action β on \mathbb{T}^2 , the \mathbb{Z}_i -action ρ on \mathbb{R}^2 , and the \mathbb{R}^2 -action α on \mathbb{T}^2 , satisfy the hypothesis of the theorem.

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4-sphere S^4 in \mathbb{R}^5 centered at (0,0,0,0,1/2) and of diameter 1, i.e.,

$$\left\{(x_1,\cdots,x_5)|x_1^2+x_2^2+x_3^2+x_4^2+\left(x_5-\frac{1}{2}\right)^2=\frac{1}{4}\right\}$$

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Define a \mathbb{T}^2 -action on S^4 by

$$\begin{pmatrix} (\theta_1, \theta_2), (x_1, \cdots, x_5) \end{pmatrix} \longrightarrow \\ (x_1, \cdots, x_5) \begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 & 0 & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 & 0 & 0 \\ 0 & 0 & \cos(\theta_2) & \sin(\theta_2) & 0 \\ 0 & 0 & -\sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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Strict Deformation θ deformation Applications

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The same formula defines also an \mathbb{R}^2 -action α on S^4 .

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$$\mathbb{Z}_2$$
-action β on \mathbb{S}^4 by reflection

$$(\sigma_2, (\mathbf{x}_1, \cdots, \mathbf{x}_5)) \longrightarrow (\mathbf{x}_1, -\mathbf{x}_2, \mathbf{x}_3, -\mathbf{x}_4, \mathbf{x}_5).$$

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 $\bullet~\mathbb{Z}_2$ also acts on \mathbb{R}^2 by reflection

$$\rho:\sigma_2\longrightarrow \left(\begin{array}{cc}-1&0\\0&-1\end{array}\right).$$

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 $\begin{array}{c} \mbox{Strict Deformation} \\ \mbox{Our work} \\ \mbox{Applications} \\ \end{array} \qquad \begin{array}{c} \mbox{Noncommutative toroidal orbifolds} \\ \mbox{θ deformation} \\ \end{array}$

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$$(\sigma_2, (\mathbf{x}_1, \cdots, \mathbf{x}_5)) \longrightarrow (\mathbf{x}_1, -\mathbf{x}_2, \mathbf{x}_3, -\mathbf{x}_4, \mathbf{x}_5).$$

 $\bullet~\mathbb{Z}_2$ also acts on \mathbb{R}^2 by reflection

$$\rho:\sigma_2\longrightarrow \left(\begin{array}{cc}-1&0\\0&-1\end{array}\right).$$

- $J = \theta dx_1 \wedge dx_2$ on \mathbb{R}^2 .
- α, β, ρ satisfy the conditions of the theorem.

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 $\begin{array}{c} \mbox{Strict Deformation} \\ \mbox{Our work} \\ \mbox{Applications} \\ \end{array} \qquad \begin{array}{c} \mbox{Noncommutative toroidal orbifolds} \\ \mbox{θ deformation} \\ \end{array}$

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- C(S⁴) → C(S⁴_θ)(depends on J and α) = θ-deformation introduced by Connes and Landi(2000).

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Noncommutative toroidal orbifol θ deformation

• \mathbb{Z}_2 -action on $C(S_{\theta}^4)$ is strongly continuous.

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- \mathbb{Z}_2 -action on $C(S_{\theta}^4)$ is strongly continuous.
- Therefore

$$K_{ullet}(C(S^4) \rtimes \mathbb{Z}_2) = K_{ullet}(C(S^4_{\theta}) \rtimes \mathbb{Z}_2).$$

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 $\begin{array}{c} \text{Strict Deformation} \\ \text{Our work} \\ \text{Applications} \end{array} \quad \begin{array}{c} \text{Noncommutative toroidal orbifolds} \\ \theta \text{ deformation} \end{array}$

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$$K_0(C(S^4_{ heta}) \rtimes \mathbb{Z}_2) = \mathbb{Z}^4, \qquad K_1(C(S^4_{ heta}) \rtimes \mathbb{Z}_2) = 0.$$

Remark: in the above process, Z₂ is not essential, the same method works for K_•(C[∞](S⁴_θ) × Z_i), i = 3, 4, 6.

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Noncommutative toroidal orbifolds θ deformation

Thanks! 谢谢!

Yi-Jun Yao On K-theory of some Noncommutative Orbifold(joint work with Xia