On K-theory of some Noncommutative Orbifold(joint work with Xiang Tang)

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Operator Spaces, Quantum Probability and Applications
Wuhan, June 5th, 2012

1. Strict Deformation

2. Our work

3. Applications
   - Noncommutative toroidal orbifolds
   - θ deformation

Herman Weyl(1885 – 1955)

Weyl product

$f, g \in \mathcal{S}(\mathbb{R}^2)$,

$$(f *^W g)(x,y) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x+u_1, y+u_2)g(x+v_1, y+v_2) \\ \mathrm{e}^{2\pi \mathrm{i}(u_1 v_2 - u_2 v_1)}\, \mathrm{d}^2 u \mathrm{d}^2 v.$$

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Associative noncommutative product.

Marc Rieffel(1937 –)

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- we complete it into a C^*-algebra $\rightsquigarrow$ strict deformation.
- K-theory of the deformed algebra is the same as the original one.

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- $\theta \notin \mathbb{Q}$, $A_\theta = C(S^1) \rtimes_\theta \mathbb{Z}$, irrational rotation algebra.

The question

Assume on the C^*-algebra A there is a strongly continuous action α of $\mathbb{R}^n$, plus a strongly continuous action β of a compact group G, then what would be the K-theory of the ("deformed") algebra ?

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- "$[\alpha, \beta] = 0$" $\Rightarrow \beta \xrightarrow{\text{lift}}$ strongly continuous action $\tilde{\beta}$ on A_J, $A_J \rtimes_{\tilde{\beta}} G \simeq (A \rtimes_\beta G)_J$.
- $K_\bullet(A \rtimes_\beta G) = K_\bullet((A \rtimes_\beta G)_J) = K_\bullet(A_J \rtimes_{\tilde{\beta}} G)$.

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- α (resp. β) : action of $\mathbb{R}^{2n}$ (resp. $\mathbb{Z}_2$) on A, dual to its action on $\mathbb{T}^{2n}$.
- ρ: natural inclusion $\mathbb{Z}_2 \hookrightarrow SL_{2n}(\mathbb{R}, J)$, we have

$$\beta_g \alpha_x = \alpha_{\rho_g(x)} \beta_g, \quad \text{for all } g \in G, x \in \mathbb{R}^n.$$

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- i.e., the G-action β on A_J is still well-defined.
- Therefore we can consider the crossed product algebra $A_J \rtimes_\beta G$.

Main Result

Theorem (X.Tang-Y.)

When A is a separable C^*-algebra, and if the actions α, β and the group homomorphism ρ satisfy

$$\beta_g \alpha_x = \alpha_{\rho_g(x)} \beta_g, \quad \text{for any } g \in G, x \in \mathbb{R}^n.$$

Then

$$K_\bullet(A_J \rtimes_\beta G) \cong K_\bullet(A \rtimes_\beta G), \quad \bullet = 0, 1.$$

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- By Morita equivalence, we have

$$K_\bullet(A_J \rtimes_\beta G) \cong K_\bullet((\overline{\mathcal{S}}_J^A \rtimes_\nu \mathbb{R}^n) \rtimes_{\overline{\beta}} G).$$

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Theorem

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$$\begin{aligned} K_\bullet(((A \otimes \mathbb{C}_n) \rtimes_\alpha \mathbb{R}^n) \rtimes_\beta G) &\cong K_\bullet^G((A \otimes \mathbb{C}_n) \rtimes_\alpha \mathbb{R}^n) \cong K_\bullet^G(A) \\ &\cong K_\bullet(A \rtimes_\beta G), \end{aligned}$$

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- We obtain: $K_\bullet((\overline{\mathcal{S}}_J^A \otimes \mathbb{C}_n) \rtimes_\beta G) \cong K_{\bullet+n}((\overline{\mathcal{S}}_J^A \rtimes_\nu \mathbb{R}^n) \rtimes_{\overline{\beta}} G)$.

Gennadi Kasparov(1948 –)

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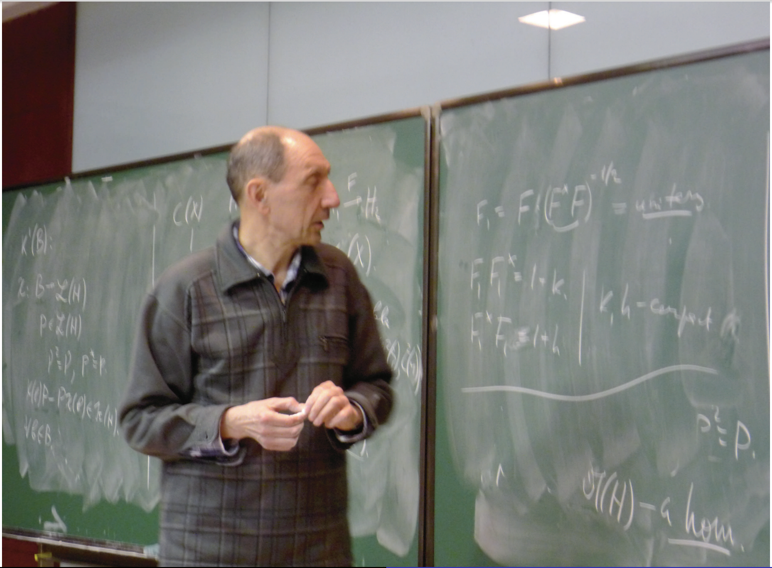
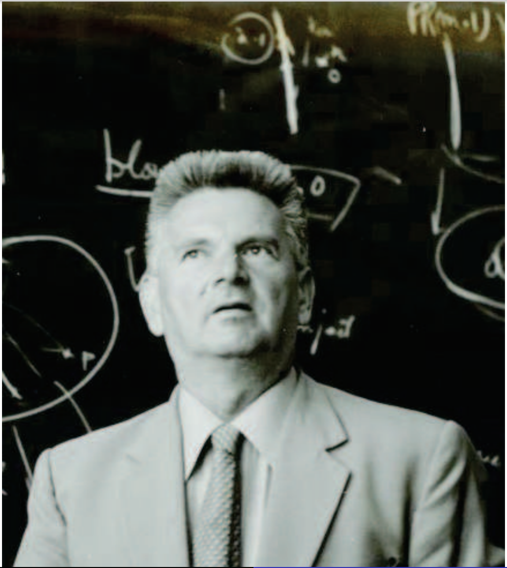


René Thom(1923 – 2002)



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- (Rieffel)For $A = \mathbb{C}$, $\overline{\mathcal{S}}_J^{\mathbb{C}}$ = space of compact operators on the subspace $\mathcal{H}$ of $L^2(U)$ generated by the elements of the form

$$g(\bar{z})e^{-\frac{\|z\|^2}{2}},$$

  where g is an anti-holomorphic function.
- $\mathcal{H}$ is a G-invariant subspace, the above isomorphism is also G-equivariant.

Third part of the proof II

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- G-equivriant Thom isomorphism $\Rightarrow$

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K_\bullet((\overline{\mathcal{S}}_J^A \otimes \mathbb{C}_n) \rtimes_{\bar\beta} G) &= K_\bullet((A \otimes \mathcal{K} \otimes C_\infty(V_0) \otimes \mathbb{C}_n) \rtimes_{\bar\beta} G) \\
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Conclusion

$$K_\bullet(A_J \rtimes_\beta G) \overset{\text{1st step}}{=\!=\!=} K_\bullet((\overline{\mathcal{S}}_J^A \rtimes_\nu \mathbb{R}^n) \rtimes_{\bar{\beta}} G)$$

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- We can define the inclusion $\rho : \mathbb{Z}_i \to SL(2,\mathbb{R})$.

It is easy to verify that the $\mathbb{Z}_i$-action β on $\mathbb{T}^2$, the $\mathbb{Z}_i$-action ρ on $\mathbb{R}^2$, and the $\mathbb{R}^2$-action α on $\mathbb{T}^2$, satisfy the hypothesis of the theorem.

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We obtain then a completely different proof of a result of Echterhoff-Lück-Philipps-Walter. For the $\mathbb{Z}_2$ case, it was first done by Kumjian (1990).

4-sphere S^4 in $\mathbb{R}^5$ centered at $(0,0,0,0,1/2)$ and of diameter 1, i.e.,

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Define a $\mathbb{T}^2$-action on S^4 by

$$((\theta_1,\theta_2),(x_1,\cdots,x_5)) \longrightarrow$$

$$(x_1,\cdots,x_5)\begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 & 0 & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 & 0 & 0 \\ 0 & 0 & \cos(\theta_2) & \sin(\theta_2) & 0 \\ 0 & 0 & -\sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

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The same formula defines also an $\mathbb{R}^2$-action α on S^4.

- $\mathbb{Z}_2$-action β on S^4 by reflection

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- $J = \theta dx_1 \wedge dx_2$ on $\mathbb{R}^2$.
- α, β, ρ satisfy the conditions of the theorem.
- $C(S^4) \rightsquigarrow C(S^4_\theta)$(depends on J and α) $= \theta$-deformation introduced by Connes and Landi(2000).

- $\mathbb{Z}_2-$action on $C(S^4_\theta)$ is strongly continuous.

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- Remark: in the above process, $\mathbb{Z}_2$ is not essential, the same method works for $K_\bullet(C^\infty(S^4_\theta) \rtimes \mathbb{Z}_i)$,$i = 3, 4, 6$.

Thanks! 谢谢!