Spectral Multipliers for Operators with Generalized Gaussian Estimates

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Abstract: In this talk I will describe some recent results on spectral multipliers for abstract self-adjoint operators with generalized Gaussian estimates.

- Joint work with
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  - ◇ El Maati Ouhabaz (Université Bordeaux 1, France)
  - ◇ Adam Sikora (Macquarie University, Australia)

Background

Fourier Analysis

- Fourier transform

$$S_R(f)(x) = \int_{|\xi|<R} \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

where

$$\hat{f}(\xi) = L^2 - \lim_{R\to\infty} \int_{|x|<R} f(x) e^{-2\pi i x\cdot\xi} dx$$

- If $f \in L^2(\mathbb{R}^n)$, then

$$\lim_{m\to\infty} S_R(f)(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n \ ?$$

(Lusin conjecture, OPEN!)

$\Rightarrow$ Progress in L^p-norm

- Fourier analysis

$$f(x) = \lim_{R\to\infty} \int_{|\xi|<R} \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

◇ $n = 1$: holds in L^p norm, $1 < p < \infty$

◇ $n \geq 2$: only when $p = 2$

- Bochner-Riesz summability

$$S_R^\delta f(x) = \lim_{R\to\infty} \int_{|\xi|<R} \left(1 - \frac{|\xi|^2}{R}\right)^\delta \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

◇ L^p-boundedness?

- Fourier multipliers

$$u(x,t) = \int_{\mathbb{R}^n} m_t(|\xi|)\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

where $m_t(|\xi|)$ is

◇ $e^{-t|\xi|^2}$ (Heat equation)

◇ $e^{-t|\xi|}$ (Laplace equation)

◇ $\cos t|\xi|$ or $\sin t|\xi|/|\xi|$ (Wave equation)

◇ $e^{-it|\xi|^2}$ (Schrödinger equation)

- L^p-boundedness of multipliers?

The Classical Case: $\mathbb{R}^n$

Radial Fourier Multipliers

- Marcinkiewicz (1939)
  Mihlin (1957)
  Hörmander (1960)

- Let $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ be the Laplacian on $\mathbb{R}^n$, and consider the operator $T_m = m(\sqrt{-\Delta})$, i.e.

$$T_m(f)^{\wedge}(\xi) = m(|\xi|)\hat{f}(\xi)$$

$\Rightarrow$

$$\|T_m(f)\|_p \leq C\|f\|_p, \quad 1 < p < \infty \ ?$$

$\diamond$ If $m \in L^\infty$, then T_m is bounded on $L^2(\mathbb{R}^n)$

Multiplier Theorem (I)

- Theorem A $T_m = m(\sqrt{-\Delta})$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if

$$|m^{(j)}(\lambda)| \leq A\lambda^{-j}, \quad 0 \leq j \leq k, \quad k > n/2 \tag{0.1}$$

or more generally:

$$\sup_{t>0} \|\eta(\cdot)m(t\cdot)\|_{W^{\alpha,2}} < \infty \qquad \alpha > n/2, \tag{0.2}$$

where $\eta \in C_0^\infty(\mathbb{R}_+)$

◇ Calderón-Zygmund Theory: (0.2) $\Rightarrow T_m f = K * f$ with

$$\int_{|x|\geq 2|y|} |K(x-y) - K(x)|dx \leq A$$

- The sharp restriction $\alpha > n/2$ in the theorem

◇ Bochner-Riesz means: $m(\lambda) = (1-\lambda^2)_+^\delta$

$$\delta > \frac{n-1}{2} \Longleftrightarrow m \in W^{\alpha,2}(\mathbb{R}), \quad \alpha > n/2$$

◇ Fix $p \in (1,2)$. Find minimal smoothness conditions for L^p-boundedness of multipliers $m(\sqrt{-\Delta})$?

L^p-Multiplier Theorem (II)

- Theorem B Let $1 < p \leq 2(n+1)/(n+3)$. $T_m = m(\sqrt{-\Delta})$ is bounded on $L^p(\mathbb{R}^n)$, if

$$\sup_{t>0} \|\eta(\cdot)m(t\cdot)\|_{W^{\alpha,2}} < \infty \qquad \alpha > \delta(p) + 1/2$$

where

$$\delta(p) = n|1/2 - 1/p| - 1/2$$

◇ Carbery-Gasper-Trebels (1984), Christ (1985), Seeger (1986),...

◇ Heo-Nazarov-Seeger (2011)

Basic Fact

$$m(\sqrt{-\Delta}) = \int_0^\infty m(\lambda) dE_{\sqrt{-\Delta}}(\lambda)$$

- The spectral measure $dE_{\sqrt{-\Delta}}(\lambda)$ satisfies

$$(R_p) \quad \left\| dE_{\sqrt{-\Delta}}(\lambda) \right\|_{L^p \to L^{p'}} \le C \lambda^{n(\frac{2}{p}-1)-1}, \quad p \in \left[1, \tfrac{2(n+1)}{n+3}\right]$$

where

$$dE_{\sqrt{-\Delta}}(\lambda; x, y) = \frac{\lambda^{n-1}}{(2\pi)^n} \int_{|\xi|=1} e^{i(x-y)\cdot\lambda\xi} d\xi$$

$\Rightarrow L^p$-Multiplier Theorem

Application: Bochner-Riesz Means

- Known results
  - ◇ $\delta > \frac{n-1}{2}$
    bounded on $L^p(\mathbb{R}^n), 1 \le p \le \infty$
  - ◇ $\delta = \frac{n-1}{2}$ (M. Christ, 1988)
    weak type $(1,1)$
  - ◇ $\delta = 0$ (C. Fefferman, 1971)
    never bounded on $L^p(\mathbb{R}^n)$ unless $n = 1$ or $p = 2$
  - ◇ $\delta < \frac{n-1}{2}$ (C. Herz, 1954)
    If bounded on $L^p(\mathbb{R}^n)$, then $\delta > \delta(p) = n\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}$, i.e.,

$$p_0(\delta) = \frac{2n}{n+1+2\delta} < p < \frac{2n}{n-1-2\delta} = p_0^{'}(\delta)$$

◇ $n = 2$ (Carleson and Sjolin, 1972)

$$p_0(\delta) < p < p_0^{'}(\delta)$$

◇ $n \geq 3$ (Stein-Tomas(1975);Fefferman(1973);Christ(1988)) bounded on $L^p(\mathbb{R}^n)$, if

$$p_0(\delta) < p < p_0^{'}(\delta)$$

and $\left|\frac{1}{p} - \frac{1}{2}\right| \geq \frac{1}{n+1}$, i.e.,

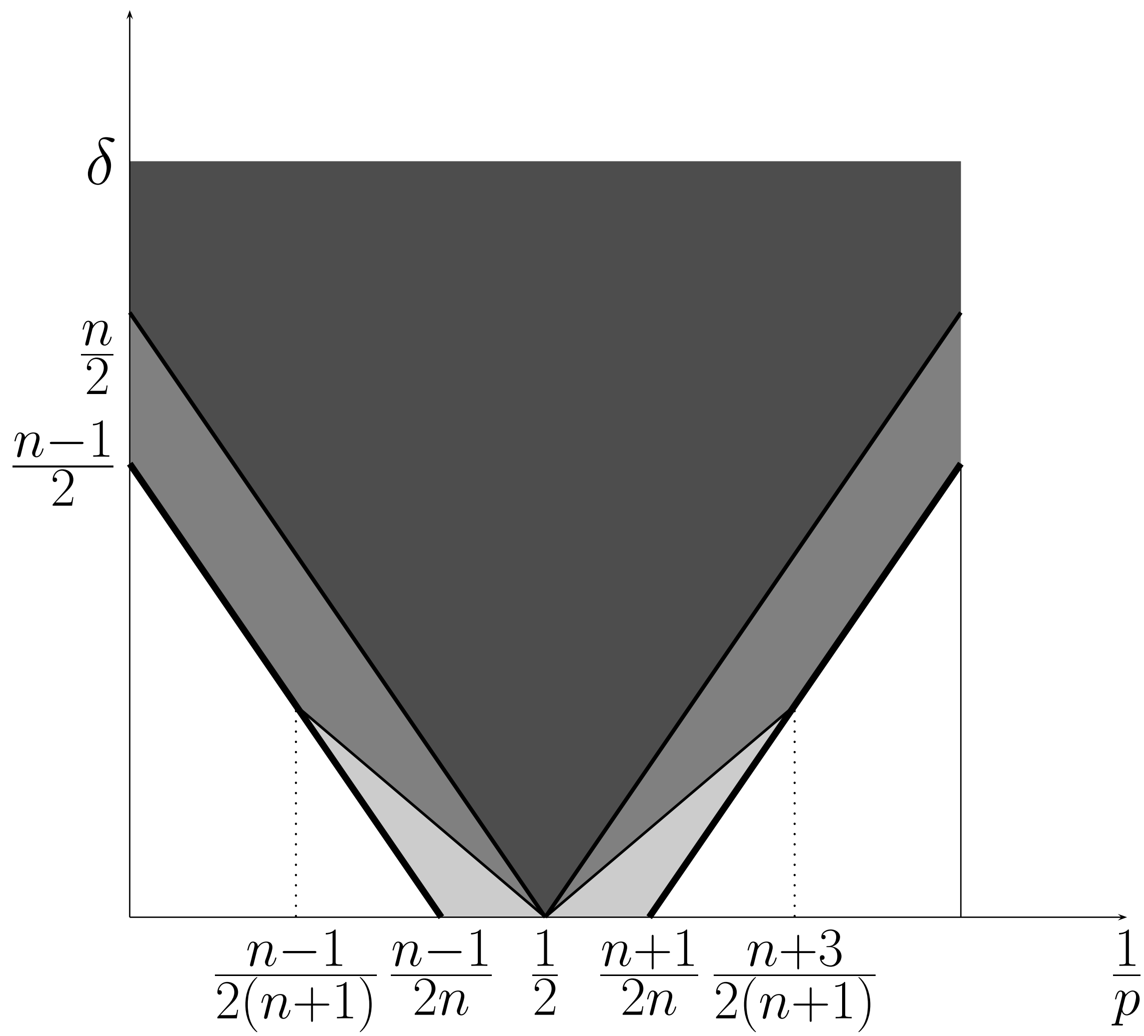
$$1 \leq p \leq \frac{2(n+1)}{n+3} \quad \text{or} \quad \frac{2(n+1)}{n-1} \leq p \leq \infty$$

....

◇ Bourgain-Guth (2011)

Bochner-Riesz summability

S_λ^δ is uniformly bounded on L^p in the shaded region.



Spectral Multipliers

- Let $L = a(D) = \sum_{|j|\le m} a_j D^j$ be a positive self-adjoint operator on $L^2(\mathbb{R}^n)$.

$$L = \int_0^\infty \lambda \, dE_L(\lambda)$$

◇ $\lim_{\lambda\to\infty} E_L(\lambda) f = f$ in $L^2(\mathbb{R}^n)$

◇ Not True in $L^p(\mathbb{R}^n), 1 \le p < 2$

◇ $E_L(\lambda)$ is a convolution operator with the kernel

$$e(\lambda; x, y) = (2\pi)^{-n} \int_{a(\xi)\le\lambda} e^{i(x-y)\cdot\xi} d\xi.$$

(Carleman (1935), Garding (1953), Agmon (1967), Hörmander (1968), ...)

- Abel-Laplace summability

$$E_\lambda^L = \int_0^\infty e^{-\mu/\lambda} dE_L(\mu)$$

◇ $\lim_{\lambda\to\infty} E_\lambda^L f = f$ in $L^p(\mathbb{R}^n), 1 \le p \le 2$

- Riesz summability

$$S_\lambda^\delta = \int_0^\infty \left(1 - \frac{\mu}{\lambda}\right)_+^\delta dE_L(\mu)$$

◇ For any $\delta > 0$, $\lim_{\lambda\to\infty} E_\lambda^L f = f$ in $L^2(\mathbb{R}^n)$.

◇ How about $L^p(\mathbb{R}^n), 1 \le p < \infty$?

- Let L be a non-negative self-adjoint operator on $L^2(X)$ and $E_L(\lambda)$ denote a spectral resolution. Define, for any bounded Borel function F,

$$F(L) = \int_0^\infty F(\lambda) dE_L(\lambda)$$

$\Rightarrow$

$$\|F(L)f\|_p \leq C\|f\|_p, \quad 1 < p < \infty \ ?$$

$\diamond$ $F(L)$ is bounded on $L^2(X)$

$\diamond$ How about $L^p(\mathbb{R}^n), 1 \leq p < \infty$?

Example: $L = \Delta + x^2$

- Riesz means $S_R^\delta(L)$: Different from $S_R^\delta(\Delta)$

  (Askey-Wainger (1965); Thangavelu (1989))

  ◇ never bounded on $L^1(\mathbb{R})$ unless $\delta > \frac{1}{6}$

  ◇ $\delta > \frac{1}{6}$
  bounded on $L^p(\mathbb{R}), 1 \leq p \leq \infty$

  ◇ $\delta < \frac{1}{6}$
  bounded on $L^p(\mathbb{R})$, if

$$\frac{4}{6\delta + 3} < p < \frac{4}{1 - 6\delta}$$

Spectral Multipliers on Compact Manifolds

- Hörmander (1966), ...

  Sogge (1987), Christ-Sogge (1988), Seeger-Sogge (1989), Tao (1996), ...

- Let P be a first order classical pseudo-differential operator on a compact manifold. Let $\{\lambda_j\}$ and and $\{e_j\}$ be the eigenvalues and eigenfunctions of P. Define,

$$m(P)f = \sum_{j=1}^{\infty} m(\lambda_j)e_j(f)$$

$\Rightarrow$

$$\|m(P)f\|_p \leq C\|f\|_p, \quad 1 < p < \infty \, ?$$

◇ If $m \in L^\infty$, then $m(P)$ is bounded on $L^2(M)$

L^p-Multiplier Theorem (III)

- Theorem C (Seeger-Sogge, 1989) Let P be such that the co-spheres $\{\xi \in T_x^*M \backslash 0 : p(x,\xi) = 1\}$ have everywhere non-vanishing Gaussian curvature for each $x \in M$. Let $1 < p \leq 2(n+1)/(n+3)$. Then $m(P)$ is bounded on $L^p(\mathbb{R}^n)$, if

$$\sup_{t>0} \|\eta(\cdot)m(t\cdot)\|_{W^{\alpha,2}} < \infty \qquad \alpha > \delta(p) + 1/2$$

where

$$\delta(p) = n|1/2 - 1/p| - 1/2$$

- Two key ingredients of the proofs
  - ◇ (discrete) $(p,2)$ restriction theorem (Sogge, 1988): Let

$$\chi_k f = \sum_{\lambda_j \in [k-1,k]} e_j(f)$$

$\Rightarrow$

$$\|\chi_k f\|_{L^2(M)} \leq C k^{\delta(p)} \|f\|_{L^p(M)}, \quad k = 1, 2, \cdots$$

$\Rightarrow$ Local multiplier theorem

  - ◇ L^p-multiplier theorem

(Littman-McCarthy-Riviere (1968); Carbery (1986); Seeger (1988))

Application: Riesz Means

- Riesz means on compact manifolds

$$S_R^\delta(f) = \sum_{j=1}^{\infty} \left(1 - \frac{\lambda_j}{R}\right)_+^\delta e_j(f)$$

◇ Let $1 \le p \le 2(n+1)/(n+3)$. If $\delta(p) = n\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}$, then $S_R^{\delta(p)}$ are of weak-type (p,p) uniformly in R (Christ-Sogge (1988); Seeger (1991); Tao (1996))

Spectral Multipliers on Metric Spaces

- Peetre (1964)
  Hörmander (1966)
  Thangavelu (1989)
  Hebisch (1990)
  Christ(1991)
  Alexopoulos(1994)
  Müller-Stein(1994)
  Duong-Ouhabaz-Sikora(2002)
  Guillarmou-Hassell-Sikora(2010)
  ... ...

Assumption (I)

- (X, d, μ) : metric measure space with “doubling volume property"

$$V(x, 2r) \leq CV(x, r) \quad \forall x \in X, r > 0$$

$\Rightarrow$

$\diamond$ $\exists\, n, C_n > 0$ such that

$$\frac{V(x,r)}{V(x,s)} \leq C_n \left(\frac{r}{s}\right)^n, \quad \forall\, r \geq s > 0,\ x \in X$$

Assumption (II)

- Let L be a non-negative self-adjoint operator on $L^2(X)$, and satisfy the finite speed propagation property

$$(FS) \qquad \operatorname{supp}\cos(t\sqrt{L}) \subset \{(x,y) : d(x,y) \le t\}$$

$\diamond$ (FS) $\Longleftrightarrow$ Davies-Gaffney estimates: For open subsets $E, F \subset X$,

$$\|e^{-tL}f\|_{L^2(F)} \le C\exp\Big(-\frac{\mathrm{dist}(E,F)^2}{ct}\Big)\|f\|_{L^2(E)}$$

◇ (FS) $\Leftarrow$ The heat kernel $p_t(x,y)$ of e^{-tL} satisfies

$$(GE) \qquad |p_t(x,y)| \le \frac{C}{V(x,\sqrt{t})} \exp\Big(- \frac{d(x,y)^2}{ct} \Big)$$

◇ Examples

1) $L = -\sum_{i,j} \partial_i \, a_{ij} \, \partial_j$ with $a_{ij} = a_{ji}$

2) $L = -\Delta + V \ (0 \le V \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n))$

3) the Laplace-Beltrami operator on a complete Riemannian manifold

L^p-Multiplier Theorem (IV)

- Theorem D (Guillarmou, Hassell and Sikora, 2010) Let $1 \le p < 2$ and $s > n(1/p - 1/2)$. If

$$V(x,r) \sim r^n,$$

and L satisfies restriction estimates

$$(R_p) \qquad \big\| dE(\lambda) \big\|_{L^p \to L^{p'}} \le C \lambda^{n(\frac{2}{p}-1)-1},$$

then for even $F \in W^{s,2}(\mathbb{R})$ supported in $[-1,1]$, $F(\sqrt{L})$ is bounded on $L^p(X)$, and

$$\sup_{t>0} \|F(t\sqrt{L})\|_{p \to p} \le C \|F\|_{W^{s,2}}$$

- If restriction estimates (R_p) holds for $p \in \left[1, \frac{2(n+1)}{n+3}\right]$, then for all $\delta > n\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}$,

$$\sup_{R>0} \left\| \left(I - \frac{L}{R}\right)_+^\delta \right\|_{L^p \to L^p} < \infty$$

(R_p) $\Leftarrow$ Schrödinger operators on asmptotically conic manifolds)

Two Questions

- Remove condition of "supported in $[-1,1]$ of F"?

  (Littman-McCarthy-Riviere (1968); Carbery (1986), Seeger (1988))

- Riesz means $\left(I - \frac{L}{R}\right)_+^\delta$ when $\delta = n\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}$?

  (Christ (1988); Christ-Sogge (1988); Tao (1996))

Main results and Key Tools

Assumption (III)

- L satisfies the Stein-Tomas restriction type condition ($ST^2_{p,2}$) for $1 \le p < 2$ if $\operatorname{supp} F \subset [0, R]$,

$(ST^2_{p,2})$

$$\left\|F(\sqrt{L})P_{B(x,r)}\right\|_{p\to 2} \le C V(x,r)^{\frac{1}{2}-\frac{1}{p}} \left(Rr\right)^{n(\frac{1}{p}-\frac{1}{2})} \left\|F(R\cdot)\right\|_{L^2}$$

for all $x \in X, r \ge 1/R$

$\diamond$ ($p = 1$: Duong-Ouhabaz-Sikora, 2002)

Assumption (IV)

- L satisfies the Sogge spectral cluster condition ($SC^{2,k}_{p,2}$) for $1 \le p < 2$ and $k \in \mathbb{N}$ if supp $F \subseteq [-N, N]$,

  $(SC^{2,k}_{p,2})$

  $$\left\|F(\sqrt{L})P_{B(x,r)}\right\|_{p\to 2} \le C V(x,r)^{\frac{1}{2}-\frac{1}{p}}(Nr)^{n(\frac{1}{p}-\frac{1}{2})}\|F(N\cdot)\|_{N^k,2}$$

  for all $x \in X, r \ge 1/R$

  ◇ For any F with supp $F \subseteq [-1, 2]$,

  $$\|F\|_{N,2} = \Big(\frac{1}{2N}\sum_{\ell=1-N}^{N} \sup_{\lambda\in[\frac{\ell-1}{N},\frac{\ell}{N})} |F(\lambda)|^2\Big)^{1/2}$$

  ◇ ($p = 1$: Cowling-Sikora, 2001)

Relations between Restrictions Estimates

- Proposition (Chen-Ouhabaz-Sikora-Yan, 2011)

  (i) If $V(x,r) \sim r^n$, then $(ST^2_{p,2}) \Leftrightarrow (R_p)$

  $$(R_p) \qquad \big\| dE(\lambda) \big\|_{L^p \to L^{p'}} \leq C\lambda^{n(\frac{2}{p}-1)-1}$$

  (ii) If $\mu(X) < \infty$ and $V(x,r) \sim r^n$, then $(SC^{2,2}_{p,2}) \Leftrightarrow (S_p)$

  $$(S_p) \qquad \big\| E_{\sqrt{L}}[\lambda, \lambda+1] \big\|_{L^p \to L^{p'}} \leq C(1+\lambda)^{n(\frac{2}{p}-1)-1}$$

- Theorem 1 (Chen-Ouhabaz-Sikora-Yan, 2011) If L satisfies (FS) and $(ST^2_{p,2})$ for some $p \in [1, 2)$, then for any even F such that $\sup_{t>0} \|\eta F(t\cdot)\|_{W^{s,2}} < \infty$ for some $s > n(1/p - 1/2)$,

$$\|F(\sqrt{L})\|_{r\to r} \leq C \sup_{t>0} \|\eta F(t\cdot)\|_{W^{s,2}}, \quad p < r < p'$$

◇ (Sikora-Yan, 2012)

- Theorem 2 (COSY, 2011) If L satisfies (FS), $(\mathrm{E}_{\mathrm{p},2})$ and $(SC^{2,2}_{p,2})$ for some $p \in [1,2)$, and for all even Borel functions F such that supp $F \subset [-N, N]$,

$$(\mathrm{AB_p}) \qquad \|F(\sqrt{L})\|_{p\to p} \le C_\varepsilon N^{\kappa n(\frac{1}{p}-\frac{1}{2})+\varepsilon}\|F(N\cdot)\|_{N^\kappa,q},$$

then for any even F such that $\sup_{t>0}\|\eta F(t\cdot)\|_{W^{s,\,2}} < \infty$ for some $s > n(1/p - 1/2)$,

$$\|F(\sqrt{L})\|_{r\to r} \le C \sup_{t>0}\|\eta F(t\cdot)\|_{W^{s,\,2}}, \quad p < r < p'.$$

◇ (Sikora-Yan, 2012)

$$(\mathrm{E}_{\mathrm{p},2}) \qquad \left\|e^{-t^2 L}P_{B(x,r)}\right\|_{p\to 2} \le CV(x,r)^{\frac{1}{2}-\frac{1}{p}}\left(\frac{r}{t}\right)^{n(\frac{1}{p}-\frac{1}{2})}.$$

- Proposition 1 (COSY, 2011) If $\mu(X) < \infty$, and L satisfies (FS) and $(SC^{2,2}_{p,2})$ for some $p \in [1, 2)$, then for any even F such that $\sup_{t>0} \|\eta F(t\cdot)\|_{W^{s,2}} < \infty$ for some $s > n(1/p - 1/2)$,

$$\|F(\sqrt{L})\|_{r\to r} \leq C \sup_{t>0} \|\eta F(t\cdot)\|_{W^{s,2}}, \quad p < r < p'.$$

- Theorem 3 (COSY, 2011) If L satisfies (FS) and restriction estimates $(ST^2_{p,2})$ for some $p \in [1,2)$, then

$$\sup_{R>0} \left\| \left(I - \frac{L}{R} \right)^{\delta(p)} \right\|_{L^p \to L^{p,\infty}} < \infty.$$

where

$$\delta(p) = \max\left\{ 0, n\left| \frac{1}{p} - \frac{1}{2} \right| - \frac{1}{2} \right\}$$

- Theorem 4 (COSY, 2011) If $\mu(X) < \infty$, and L satisfies (FS) and $(SC^{2,1}_{p,2})$ for some $p \in [1,2)$, then

$$\sup_{R>0} \left\| \left(I - \frac{L}{R} \right)^{\delta(p)} \right\|_{L^p \to L^{p,\infty}} < \infty$$

Key Tools

- Key ingredients

  ◇ Conditions (FS) + Restriction estimates

  $$\Downarrow$$

  Local Multiplier Theorem

  ◇ Calderón-Zygmund theory with non-smooth kernels

  $$\Downarrow$$

  L^p-Multiplier Theorem

Restriction Estimates Revisited

- L satisfies an endpoint Strichartz estimates

$$\int_{\mathbb{R}} \|e^{itL}\|^2_{\frac{2n}{n-2}} dt \le C\|f\|_2^2, \quad n > 2$$

and for $1 \le p \le 2n/(n+2)$, the smoothing property

$$\|e^{-tL}\|_{p \to \frac{2n}{n+2}} \le Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{n+2}{2n})}, \;\; \forall t > 0$$

$\Rightarrow$ (R_p) $\Rightarrow$ $(ST^2_{p,2})$

◇ Dispersive type estimates $\|e^{isL}\|_{1\to\infty} \le C|s|^{-n/2}$
$\Rightarrow$ Endpoint Strichartz estimates

Applications

- (I): Standard Laplace operator and compact manifolds

◇ Standard Laplace operator

$\Rightarrow$ (R_p) for $1 \leq p \leq 2(n+1)/(n+3)$

(Stein, 1967; Tomas, 1975)

$\Rightarrow$ Theorems 1 and 3 for $1 \leq p \leq 2(n+1)/(n+3)$

◇ Compact manifolds

$\Rightarrow$ (S_p) for $1 \leq p \leq 2(n+1)/(n+3)$

(Sogge, 1987, 2002)

$\Rightarrow$ Theorems 2 and 4 for $1 \leq p \leq 2(n+1)/(n+3)$

- (II): Asmptotically conic manifolds
  - ◇ Schrödinger operators, i.e. $\Delta + V$ on $\mathbb{R}^n$, where V smooth and decaying sufficiently at infinity
  - ◇ The Laplacian with respect to metric perturbations of the flat metric on $\mathbb{R}^n$, again decaying sufficiently at infinity
  - ◇ The Laplacian on asymptotically conic manifolds

$\Rightarrow$ ($ST^2_{p,2}$) for $1 \leq p \leq \frac{2(n+1)}{n+3}$

(Guillarmou-Hassell-Sikora, 2010)

$\Rightarrow$ Theorems 1 and 3 for $1 \leq p \leq \frac{2(n+1)}{n+3}$

- (III): Operators $\Delta_n + \frac{c}{r^2}$ acting on $L^2((0,\infty), r^{n-1}dr)$
Fix $n > 2$ and $c > -(n-2)^2/4$, write

$$Lf = (\Delta_n + \frac{c}{r^2})f = -\frac{d^2}{dr^2}f - \frac{n-1}{r}\frac{d}{dr}f + \frac{c}{r^2}f$$

$\Rightarrow$ ($ST^2_{p,2}$) for $p \in ((p_c^*)', 2n/(n+1))$ where

$$p_c^* = \frac{n}{(n-2)/2 - \sqrt{(n-2)^2/4 + c}}$$

(Chen-Ouhabaz-Sikora-Yan, 2011)

$\Rightarrow$ Theorems 1 and 3 for $p \in ((p_c^*)', 2n/(n+1))$

- (IV) Scattering operators

$$L = \Delta_3 + V(x) = -(\partial_1^2 + \partial_2^2 + \partial_3^2) + V(x)$$

where $V \geq 0$ and

$$\sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{V(x)}{|x-y|} dy < 4\pi \quad \text{and} \quad \int_{\mathbb{R}^6} \frac{V(x)V(y)}{|x-y|^2} dx dy < 4\pi^2$$

$\Rightarrow$ ($ST^2_{p,2}$) for $1 \leq p < 6/5$

(Rodnianski-Schlag, 2004)

$\Rightarrow$ Theorems 1 and 3 for $1 \leq p < 6/5$

- (V) The harmonic oscillator ($n \geq 2$)

$$L = \Delta + V(x)$$

where

$$V \sim |x|^2, \quad |\nabla V| \sim |x|, \quad |\partial_x^2 V| \leq C$$

$\Rightarrow$ ($SC_{p,2}^{2,2}$) for $1 \leq p < \frac{2n}{n+2}$

(Koch-Tataru, 2005)

$\Rightarrow$ Theorem 2 for $1 \leq p < \frac{2n}{n+2}$

- (VI): Homogeneous groups

  Spectral multipliers for the homogeneous Laplace operators acting on homogeneous groups were investigated by Hulanicki and Stein, Folland and Stein, and De Michele and Mauceri.

$$\|F(\sqrt{L})\|_{L^2\to L^\infty}^2 = C\int_0^\infty |F(t)|^2 t^{n-1}dt.$$

$\Rightarrow$ ($ST^2_{1,2}$)

$\Rightarrow$ Theorems 1 and 3 for $p=1$

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Thank You !