Close II_1 Factors and the Isomorphism Problem

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Notation

- H separable infinite dimensional Hilbert space
- $B(H)$ bounded linear operators on H
- A,B not necessarily unital C^* algebras represented concretely on H
- A_1 the unit ball of A
- $A' = \{T \in B(H) \mid ST = TS \ \forall \ S \in A\}$
- $\mathcal{U}(A)$ the unitary group of A

The Hausdorff Metric

Definition

$$d(A,B) = \max(\sup_{x\in A_1}\inf_{y\in B_1}\|x-y\|, \sup_{x\in B_1}\inf_{y\in A_1}\|x-y\|)$$

Questions (Kadison/Kastler): Is there a universal constant $c > 0$ such that

1. $d(A,B) < c \Rightarrow A$ and B are isomorphic?
2. $d(A,B) < c \Rightarrow A$ and B are spatially isomorphic?
3. $d(A,B) < c \Rightarrow A$ and B are spatially isomorphic by a unitary u with $\|u - I\| < f(c)$ where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is also universal and satisfies $\lim_{c\to 0} f(c) = 0$?

Results For C^*-Algebras

- (Barry Johnson, 1982) If $A = \{f : [0,1] \to K(H) \mid f \text{ continuous}\}$, $\exists$ arbitrarily close unitary conjugates of A such that one cannot choose the unitary close to I.
- (Choi and Christensen, 1983) $\exists$ arbitrarily close nonisomorphic nonseparable C^* algebras.
- (Christensen, Sinclair, Smith, White, Winter, 2010) Spatial isomorphism always holds whenever A is separable and nuclear and $c < 1/420,000$

Results For Von Neumann Algebras

- (Christensen, 1973, 1977) If A is an injective von Neumann algebra, then 3) holds for $c < 1/169$.
- (Christensen, 1977) If M and N II_1 factors and $(M \cup N)''$ is a finite von Neumann algebra, then 3) holds for $c < 1/8$.
- All questions still open for the class of II_1 factors

II_1 Factors: Definition and Examples

A II_1 factor is an infinite dimensional von Neumann algebra M with $\mathcal{Z}(M) := M' \cap M = \mathbb{C}I$ admitting a (unique) normal, faithful, tracial state.

Examples

1. G a discrete group with the infinite conjugacy class condition ($\mathbb{F}_n$ for $n > 1$); M is the double commutant of the left regular representation of G.
2. $(X, \mathcal{A}, \mu)$ a probability measure space, G a countable discrete group with a free, ergodic, measure-preserving action α of G on $(X, \mathcal{A}, \mu)$ where $\alpha_{gh} = \alpha_g \alpha_h$ ($G = \mathbb{Z}$ via irrational rotation, $(X, \mathcal{A}, \lambda) = (\mathbb{T}, \mathcal{B}, \lambda)$ where $\mathcal{B}$ is the σ-algebra of Borel sets and λ is Lebesgue measure). Obtain a II_1 factor $M = L^\infty(X, \mathcal{A}, \mu) \rtimes_\alpha G$, the crossed product of $(X, \mathcal{A}, \mu)$ by G.

The construction represents $M = L^\infty(X,\Omega,\mu) \rtimes_\alpha G$ on H where $H = L^2(X,\mathcal{A},\mu) \otimes \ell_2(G)) = \{f : G \to L^2(X,\mathcal{A},\mu) \mid \sum_{g\in G} \|f(g)\|^2 < \infty\}$

M is generated as a von Neumann algebra by copies $\{u_g\}_{g\in G}$ of G and $\{\pi_\alpha(x)\}_{x\in L^\infty(X,\Omega,\mu)}$ of $L^\infty(X,\Omega,\mu)$ satisfying, for all $x \in L^\infty(X,\Omega,\mu)$, $f \in H$, and $g, h \in G$

$$(\pi_\alpha(x)f)(g) = \alpha_{g^{-1}}(x)f(g)$$

$$(u_h f)(g) = f(h^{-1}g)$$

Then $u_h\pi_\alpha(x)u_{h^{-1}} = \pi_\alpha(\alpha_h(x))$

We may then consider M as sums of the form $\sum_{g\in G} \pi_\alpha(x_g)u_g$ for $x_g \in L^\infty(X,\mathcal{A},\mu)$ with multiplication defined using the above identity.

Let $G \curvearrowright_{\alpha} (X, \Omega, \mu)$. A 2-cocycle in $Z^2(G, \mathcal{U}(L^{\infty}(X, \Omega, \mu)))$ is a map $\omega : G \times G \to \mathcal{U}(L^{\infty}(X, \Omega, \mu))$ satisfying, for all $g, h, k \in G$,

$$\alpha_g(\omega(h, k))\omega(gh, k)^*\omega(g, hk)\omega(g, h)^* = I.$$

Then we may define a product $u_g u_h := \omega(g, h)u_{gh}$ and obtain the twisted crossed product $M = L^{\infty}(X, \mathcal{A}, \mu) \rtimes_{\alpha,\omega} G$.

The isomorphism type of the twisted crossed product is completely determined by the cohomology class of ω in $H^2(G, \mathcal{U}(A))$. In particular, any cocycle cohomologous to $\omega(g, h) = I$ gives $M = L^{\infty}(X, \mathcal{A}, \mu) \rtimes_{\alpha} G$

Main Theorem

Theorem

(CCSSWW) Let M be $*$-isomorphic to $L^\infty(X, \mathcal{A}, \mu) \rtimes_\alpha \mathbb{F}_n$ for some $n \geq 2$ where α is free, ergodic, and measure-preserving. Then if N is any other von Neumann algebra with $d(M, N) < 5.8 \times 10^{-16}$, N is $*$-isomorphic to M.

Let N be another von Neumann algebra with $d(N, M) < 5.8 \times 10^{-16}$. Let Q denote the copy of $L^{\infty}(X, \Omega, \mu)$ in M.

1. Conjugate N by a unitary u close to the identity to acheive $Q \subset uNu^* =: N_0$. Replace N by N_0.
2. Transfer $\{u_g\}_{g \in G}$ to unitaries $\{v_g\}_{g \in G}$ in N_0 such that $u_g x u_g^* = v_g x v_g^*$ for all $g \in G$ and $x \in Q$.
3. Find representations of M and N_0 on a new Hilbert space on which N_0 and M are still close and both are in standard form; obtain N_0 is isomorphic to $L^{\infty}(X, \mathcal{A}, \mu) \rtimes_{\alpha,\omega} G$.
4. Conclude N_0 is isomorphic to $Q \rtimes_{\alpha} \mathbb{F}_n$ by using cohomological properties of $\mathbb{F}_n$ to trivialize the cocycle.

For a von Neumann subalgebra B of M, let $\mathcal{N}_M(B)$ denote the group of unitaries u in M with $uBu^* = B$.

If $M = Q \rtimes_{\alpha,\omega} G$, then $\mathcal{N}_M(Q)'' = M$.

For any unital von Neumann subalgebra B of a II_1 factor N, $\exists$ a unique normal, faithful trace-preserving conditional expectation $\mathbb{E}_B : N \to B$.

Near Inclusions

Let $\delta \geq 0$. We write $A \subseteq_\delta B$ if $\forall x \in A$, $\exists y \in B$ with $\|x - y\| \leq \delta\|x\|$. We write $A \subset_\delta B$ if $\exists\ 0 \leq \gamma < \delta$ with $A \subseteq_\gamma B$.

Note $d(A, B) < \delta \Rightarrow A \subset_\delta B \subset_\delta A$, but $A \subset_\delta B \subset_\delta A$ only implies $d(A, B) < 2\delta$.

So at the cost of slightly worse estimates, 2-sided δ-containments can be used (better properties under taking commutants and amplifications).

Folklore

Let P be a finite von Neumann algebra. If G is a countable discrete group and $G \curvearrowright_\alpha P$ is a trace-preserving action of G by automorphisms, then if $\omega \in Z^2(G, \mathcal{U}(\mathcal{Z}(P)))$,

- Any finite von Neumann algebra M generated by a unital copy of P and unitaries $\{w_g\}_{g \in G}$ such that $P' \cap M \subseteq P$ and there exists a trace-preserving conditional expectation $\mathbb{E} : M \to P$ satisfying $w_g x w_g^* = \alpha_g(x)$, $w_g w_h = \omega(g,h) u_{gh}$, and $\mathbb{E}(w_g^* w_h) = \delta_{g,h} I$ for all $x \in M$ and $g, h \in G$, then M is $*$-isomorphic to $P \rtimes_{\alpha,\omega} G$
- If $[\omega] = [\omega']$ in $H^2(G, \mathcal{U}(\mathcal{Z}(P)))$, then $P \rtimes_{\alpha,\omega} G$ is $*$-isomorphic to $P \rtimes_{\alpha,\omega'} G$.

Lemma 1

Assume $d(M,N) < 5.8 \times 10^{-16} < \frac{1}{8}$. Then we have that N is a II_1 factor by results of Kadison and Ringrose (1972).

Lemma

(Christensen, 1980) Suppose $d(M,N) < 1/100$. If $A \subset M$ is amenable, then $\exists$ a unitary $u \in (A \cup N)''$ with $\|I - u\| \leq 150 d(M,N)$ such that $A \subseteq uNu^* := N_0$. Further, if M and N are amenable and $M, N \subseteq P$ where P is finite, $N \subset_\delta M$, $M \subset_\delta N$ for $\delta < \frac{1}{8}$, then $\exists\ u \in \mathcal{U}((M \cup N)'')$ such that $\|I - u\| \leq 6.5\delta$ and $uMu^* = N$.

By replacing N with N_0, we may assume $Q \subseteq N \cap M$.

Lemma 2

Lemma

Suppose $M \subset_\gamma N \subset_\gamma M$ *for* $0 < \gamma < \frac{1}{16\sqrt{2}}$. *Then if* $A \subset N \cap M$ *is amenable,*

- $\forall\ u \in \mathcal{N}_M(A)$, $\exists\ v \in \mathcal{N}_N(A)$ *with* $\|u - v\| < 14\sqrt{2}\gamma$.
- *If* $u \in \mathcal{N}_M(A)$, $v \in \mathcal{N}_N(A)$ *satisfying* $\|u - v\| < \frac{1}{2}$, $\exists\ w \in \mathcal{U}(A)$ *and* $w' \in \mathcal{U}(A')$ *with* $\|w - I\| < 2^{3/2}\|u - v\|$, $\|w' - I\| < (2^{3/2} + 1)\|u - v\|$ *and* $v = uw'w$.

Proof of Lemma 2, Part 1

Proof: Let $u \in \mathcal{N}_M(A)$. It is known (Koshkam, 1984) that we can find $w \in \mathcal{U}(N)$, $\|u - w\| < \sqrt{2}\gamma$. Let $x \in A_1$. Then

$$\|uxu^* - wxw^*\| \leq \|uxu^* - wxu^*\| + \|wxu^* - wxw^*\| \leq 2\|u - w\| \leq 2\sqrt{2}\gamma$$

This implies $d(A, wAw^*) < 2\sqrt{2}\gamma < \frac{1}{8}$. Therefore by Lemma 1, $\exists$ unitary $s \in (A \cup wAw^*)'' \subseteq N$ satisfying $\|I - s\| < 13\sqrt{2}\gamma$ and $swAw^*s^* = A$. Let $v = sw \in \mathcal{N}_N(A)$. Then

$$\|v - u\| = \|v - sw\| \leq \|v - w\| + \|w - sw\| < \sqrt{2}\gamma + 13\sqrt{2}\gamma = 14\sqrt{2}\gamma.$$

$\square$

Standard Form: Lemma 3

Lemma

Suppose $N \subset_\gamma M \subset_\gamma N$ for $\gamma < 4 \times 10^{-14}$. Then $\exists$ a separable Hilbert space K and representations $\pi : M \to B(K)$ and $\rho : N \to B(K)$ satisfying

- *$\pi(M)$ and $\pi(N)$ are in standard form on K and $\exists$ a common cyclic, separating tracial vector Ω.*
- *$\pi(M) \subset_\beta \rho(N) \subset_\beta \pi(M)$ with $\beta < 76371\gamma^{1/2} < 1/62$*
- *$\pi|_Q = \rho|_Q$*
- *$\rho(Q)' \cap \rho(N) = \rho(Q)$ and $\rho(Q)$ is regular in $\rho(N)$ (implies Q is regular in N)*
- *$\langle \pi(M), e_{\pi(Q)} \rangle = \langle \rho(N), e_{\rho(Q)} \rangle$.*
- *If $\{s_g\}_{g \in G}$ is a family of unitaries in N with $\|u_g - s_g\| < 1/2 - \beta$, then $\{s_g\}_{g \in G}$ satisfies $\mathbb{E}_Q\left(s_g^* s_h\right) = \delta_{g,h} I$ and $\bigoplus_{g \in G}(s_g Q\Omega)$ is dense in K.*

Conclusion

Note that for the unitaries from Lemma 2,

$$\|u_g - s_g\| < 7869d(M,N) < 7869(5.3 \times 10^{-16}) < 1/2 - 1/62,$$

and so the hypotheses of Lemma 3 are satisfied.

By the folklore result, $\exists$ 2-cocycle $\omega \in \mathcal{Z}(G, \mathcal{U}(Q))$ with N isomorphic to $Q \rtimes_{\alpha,\omega} \mathbb{F}_n$. However, since H^2 is trivial for all free groups, we have that ω is cohomologous to the trivial cocycle, so N is $*$-isomorphic to $Q \rtimes_\alpha \mathbb{F}_n$

Other results

- Structural properties preserved for small enough c: unique Cartan masa, strong solidity, property Γ, McDuff.
- If in addition we assume action α is not strongly ergodic, then 2) holds for $c < 5.8 \times 10^{-16}$ (Property Γ). Don't know about 3)...
- Other crossed product constructions yield 3) after tensoring with hyperfinite II_1 factor (McDuff).
- The main ingredient is the notion of $c.b.$ distance:

$$d_{cb} = \sup_{n \geq 1}(A \otimes M_n(\mathbb{C}), B \otimes M_n(\mathbb{C})).$$