Noncommutative Poisson Boundaries over Locally Compact Quantum Groups

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μ-Harmonic Functions and Poisson Boundary

Locally Compact Groups

Let G be a locally compact group. Then there is a natural multiplication on the space $M(G) = C_0(G)^*$ of bounded regular measures on G. The multiplication is defined by

$$\langle f, \mu \star \nu \rangle = \int_G \int_G h(st) d\mu(s) d\nu(t)$$

for all $h \in C_0(G)$.

$L_1(G)$ with the convolution multiplication

$$f \star g(t) = \int_G f(s) g(s^{-1}t) ds$$

is a norm closed two-sided ideal in $M(G)$.

Therefore for each $\mu \in M(G)$, we can define a right multiplication map

$$m_\mu : f \in L_1(G) \to f \star \mu \in L_1(G),$$

on $L_1(G)$.

μ-Harmonic Functions

Let G be a locally compact group and $\mu \in M(G)$ be a regular probability measure on G. We can obtain a unital completely positive (ucp) map Φ_μ on $L_\infty(G)$ given by

$$\Phi_\mu(h)(s) = \int_G h(st)d\mu(t)$$

for all $h \in L_\infty(G)$.

Since for any $f \in L_1(G)$, we have

$$\langle \Phi_\mu(h), f \rangle = \int \Phi_\mu(h)(s)f(s)ds = \int_G h(st)f(s)dsd\mu(t) = \langle h, f \star \mu \rangle,$$

we see that $\Phi_\mu = (m_\mu)^*$ is the adjoint map of the right multiplication map m_μ and thus is weak* continuous on $L_\infty(G)$. In this case, we also say that Φ_μ is a Markov operator on $L_\infty(G)$.

A function $h \in L_\infty(G)$ (or bounded Borel function) is called μ-harmonic (or Φ_μ-harmonic) if

$$\Phi_\mu(h) = h.$$

Poisson Boundary

We let

$$\mathcal{H}_\mu = \{h \in L_\infty(G) : \Phi_\mu(h) = h\}$$

be the space of all μ-harmonic functions on G. This is a weak* closed operator system in $L_\infty(G)$.

It is important to note that there is a ucp map

$$\mathcal{E} : L_\infty(G) \to \mathcal{H}_\mu \subseteq L_\infty(G)$$

from $L_\infty(G)$ onto $\mathcal{H}_\mu$ given by the weak* Banach limit

$$\mathcal{E}(h) = \lim_B \Phi_\mu^n(h)$$

for all $h \in L_\infty(G)$.

Since we are taking the Banach limit, we get $\Phi_\mu \circ \mathcal{E} = \mathcal{E}$ and thus

$$\mathcal{E}^2 = \mathcal{E}.$$

Therefore, we say that $\mathcal{E}$ is a conditional expectation from $L_\infty(\mathbb{G})$ onto $\mathcal{H}_\mu$.

Then we can obtain a von Neumann algebra multiplication on $\mathcal{H}_\mu$ given by the Choi-Effros product

$$h \circ k = \mathcal{E}(hk),$$

and we obtain a commutative von Neumann algebra $(\mathcal{H}_\mu, \circ)$, the Poisson boundary of (G, μ).

Remark: In general, $(\mathcal{H}_\mu, \circ)$ is not necessary a von Neumann subalgebra of $L_\infty(G)$.

It is known that the product on $\mathcal{H}_\mu$ coincides with the product on $L_\infty(G)$ if and only if every bounded continuous function $h \in \mathcal{H}_\mu$ must be constant on the coset of the closed subgroup G_μ generated by the support supp μ of μ.

Proposition: Let μ be a non-degenerate probability measure, i.e. the subgroup generated by the support of μ is dense in G. TFAE:

1. $(\mathcal{H}_\mu, \circ)$ is a subalgebra of $L_\infty(G)$;

2. $(\mathcal{H}_\mu, \circ) = \mathbb{C}1$.

Proposition: Let G be a locally compact group. Then TFAE:

1. There is a probability measure μ on G such that $(\mathcal{H}_\mu, \circ) = \mathbb{C}1$.

2. G is amenable and σ-compact.

Therefore, if G is a countable discrete non-amenable group, then for any non-degenerate probability measure μ on G, $(\mathcal{H}_\mu, \circ)$ can not be von Neumann sub algebra of $L_\infty(G)$.

More Properties about the Boundary

There is a natural left action $\alpha : G \curvearrowright L_\infty(G)$ given by

$$\alpha_s(h)(t) = h(s^{-1}t).$$

The Markov operator Φ_μ is invariant with respect to this action. Indeed, we have

$$\Phi_\mu \circ \alpha_s = \alpha_s \circ \Phi_\mu$$

for all $s \in G$ sicne

$$\begin{aligned} \alpha_s \circ \Phi_\mu(h)(t) &= \Phi_\mu(h)(s^{-1}t) = \int_G h(s^{-1}tg)d\mu(g) \\ &= \int_G \alpha_s(h)(tg)d\mu(g) = \Phi_\mu \circ \alpha_s(h)(t) \end{aligned}$$

for all $h \in L_\infty(G)$. Therefore, α satisfies

$$\alpha_s \circ \mathcal{E} = \mathcal{E} \circ \alpha_s$$

and thus α induces an action $\alpha_\mu : G \curvearrowright \mathcal{H}_\mu$ on the von Neumann algebra $(\mathcal{H}_\mu, \circ)$.

There exists a (unique) measure space (Ω, ν) such that

$$(\mathcal{H}_\mu, \circ) = L_\infty(\Omega, \nu)$$

and the induced action α_μ on $\mathcal{H}_\mu$ corresponds to a measure preserving action on (Ω, ν). This space (Ω, ν), where Ω can be chosen as a Borel space, gives the Poisson boundary of (G, μ).

Can we study this in duality setting ?

Yes ! Chu and Lau have considered the dual version of harmonic functions. In this case, they replaced

$L_\infty(G)$ by $VN(G)$,

and replaced

probability measures μ on G by states φ in $B(G) = C^*(G)^*$,

which are exactly positive definite functions on G with $\varphi(e) = 1$.

Given any state $\varphi \in B(G)$, we can define a multiplication map

$$m_\varphi : f \in A(G) \to \varphi f \in A(G)$$

on $A(G)$. Its adjoint map $\hat{\Phi}_\varphi = m_\varphi^*$ is a weak* continuous ucp map (i.e. a Markov operator) on $VN(G)$ such that

$$\hat{\Phi}_\varphi(\lambda_s) = \varphi(s)\lambda_s.$$

We can define φ-harmonic functionals (on $A(G)$) to be elements $x \in VN(G)$ such that $\hat{\Phi}_\varphi(x) = x$. We define the φ-Poisson boundary to be the space

$$\hat{\mathcal{H}}_\varphi = \{x \in VN(G) : \hat{\Phi}_\varphi(x) = x\} \subseteq VN(G).$$

The theory is strikingly different from the $L_\infty(G)$ case.

Theorem [Chu-Lau] For any state $\varphi \in B(G)$,

$$G_\varphi = \{g \in G : \varphi(g) = 1\}$$

is a closed subgroup of G and we have

$$\hat{\mathcal{H}}_\varphi = \lambda(G_\varphi)''$$

which is always a von Neumann subalgebra of $VN(G)$!

Can we generalize this to LCQGs ?

Poisson boundaries for qunatum measures on some discrete quantum groups have been stuided by

- Izumi 2002: $\mathbb{G} = \widehat{SU_q(2)}$, $\mathcal{H}_\mu = L_\infty(SU_q(2)/\mathbb{T})$.
- Neshveyev-Tuset 2006: $\mathbb{G} = \widehat{SU_q(N)}$, $\mathcal{H}_\mu = L_\infty(SU_q(N)/\mathbb{T}^{N-1})$.
- Vaes-Vander Vennet 2008: $\mathbb{G} = \widehat{A_o(F)}$.
- Vaes-Vander Vennet 2010: $\mathbb{G} = \widehat{A_u(F)}$

Our goal is to study Poisson boundaries and their properties for general LCQGs.

Locally Compact Quantum Groups

The theory of locally compact quantum groups was originated from the generalization of Pontryagin duality. It is well-known that if G is an abelian group, we can define the dual group

$$\hat{G} = \{\chi : G \to \mathbb{T} : \text{ continuous homomorphism}\}$$

and we have the Pontryagin duality

$$\hat{\hat{G}} = G.$$

The question is how to generalize this concept to non-abelian groups !

Let G be a locally compact group. Then we have a natural (dual) correspondence between

Locally Compact Space $G \Leftrightarrow$ Commutative C*-algebra $C_0(G)$

or

Measure Space $G \Leftrightarrow$ Commutative von Neumann Algebra $L_\infty(G)$

Hopf von Neumann Algebra Structure on $L_\infty(G)$

The group multiplication

$$(s,t) \in G \times G \to st \in G$$

on G induces a co-multiplication

$$\Gamma_a : f \in L_\infty(G) \to \Gamma_a(f) \in L_\infty(G)\bar{\otimes}L_\infty(G)$$

where $\Gamma_a(f)(s,t) = f(st)$.

The associativity of group multiplication implies the co-associativity of Γ_a, i.e. we have

$$(\Gamma_a \otimes \iota)\Gamma_a = (\iota \otimes \Gamma_a)\Gamma_a.$$

Indeed, we have

$$(\Gamma_a \otimes \iota)\Gamma_a(f)(s,t,u) = f((st)u) = f(s(tu)) = (\iota \otimes \Gamma_a)\Gamma_a(f)(s,t,u).$$

Therefore, $(L_\infty(G), \Gamma_a)$ is a commutative Hopf von Neumann algebra.

The inverse of G determines the co-inverse κ on $L_\infty(G)$, which is given by

$$\kappa(f)(t) = f(t^{-1})$$

Finally, the left Haar measure on G determines a left Haar weight

$$\varphi_a(h) = \int_G h(s)ds$$

on $L_\infty(G)^+$.

Kac Algebras

G. Kac made a very important contribution to this problem during 1960's-70's. He first introduced *Kac algebras* $\mathbb{K} = (M, \Gamma, \kappa, \varphi)$, for unimodular case, in the 60's.

The theory was completed for general (non-unimodular) case in the 70's by two groups: Kac-Vainerman in Ukraine and Enock-Schwartz in France (see Enock-Schwartz's book 1992).

Given a Kac algebra $\mathbb{K}$, we can obtain the dual Kac algebra $\hat{\mathbb{K}}$ and obtain a perfect Pontryagin duality

$$\hat{\hat{\mathbb{K}}} = \mathbb{K}.$$

Locally Compact Quantum Groups (LCQG)

The notion of *quantum groups* was introduced by Drinfel'd in his 1986 ICM talk. Here, we consider the analysis aspect of quantum groups, i,e we consider the quantization of locally compact groups.

In 1987, Woronowicz discovered $SU_q(2, \mathbb{C})$, a natural quantum deformation of $SU(2, \mathbb{C})$. He showed that $SU_q(2, \mathbb{C})$ does not correspond to any Kac algebra due to the missing of bounded co-involution.

Since then, several different definitions of LCQG have been given by

- Baaj and Skandalis 1993: Regular Multiplicative Unitaries
- Woronowicz 1996: Manageable Multiplicative Unitaries
- Kustermans and Vaes 2000: Quantum Groups, C^*-algebra setting
- Kustermans and Vaes: 2003: Quantum Groups, von Neumann algebra setting.

Kustermans and Vaes' Definition of LCQG

A *LCQG* is $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ consisting of

(1) a von Neumann algebra M, which will be denoted by $L_\infty(\mathbb{G})$

(2) a *co-multiplication* $\Gamma : M \to M \bar{\otimes} M$, i.e. a unital normal *-homomorphism satisfying the co-associativity condition

$$(id \otimes \Gamma) \circ \Gamma = (\Gamma \otimes id) \circ \Gamma.$$

(3) a left Haar wight φ, i.e. a n.f.s weight φ on M satisfying

$$(\iota \otimes \varphi)\Gamma(x) = \varphi(x)1$$

(4) a right Haar weight ψ, i.e. n.f.s weight ψ on M satisfying

$$(\psi \otimes \iota)\Gamma(x) = \psi(x)1.$$

It is known that for every locally compact quantum group $\mathbb{G} = (M, \Gamma, \varphi, \psi)$, there exists a *dual quantum group* $\widehat{\mathbb{G}} = (\widehat{M}, \widehat{\Gamma}, \widehat{\varphi}, \widehat{\psi})$ such that we may obtain the perfect Pontryagin duality

$$\hat{\hat{\mathbb{G}}} = \mathbb{G}.$$

Commutative LCQG

Let G be a locally compact group. There exists a natural co-multiplication

$$\Gamma_a : f \in L_\infty(G) \to \Gamma_a(f) \in L_\infty(G)\bar{\otimes}L_\infty(G)$$

given by

$$\Gamma_a(f)(s,t) = f(st).$$

It is easy to see the Γ_a is a normal injective *-homomorphism such that it satisfies the co-associativity condition

$$(\Gamma_a \otimes \iota)\Gamma_a = (\iota \otimes \Gamma_a)\Gamma_a$$

i.e. we have

$$(\Gamma_a \otimes \iota)\Gamma_a(f)(s,t,u) = f((st)u) = f(s(tu)) = (\iota \otimes \Gamma_a)\Gamma_a(f)(s,t,u).$$

There is a left Haar weight $\varphi_a : L_\infty(G)^+ \to [0,\infty]$ given by the integration

$$\varphi_a(h) = \int_G h(s)ds$$

w.r.t. the left Haar measure on G. We can obtain the right Haar weight ψ_a by taking the integration w.r.t. the right Haar measure.

Then $\mathbb{G}_a = (L_\infty(G), \Gamma_a, \varphi_a, \psi_a)$ is a commutative LCQG.

Co-commutative LCQG

Let G be a locally compact group. There exists a natural co-associative co-multiplication

$$\Gamma_G : VN(G) \to VN(G)\bar{\otimes}VN(G)$$

on $VN(G)$ given by

$$\Gamma_G(\lambda_s) = \lambda_s \otimes \lambda_s.$$

This is co-commutative in the sense that

$$\Sigma \circ \Gamma_G = \Gamma_G.$$

Moreover, we can obtain a normal faithful (Plancherel) weight $\varphi_G = \psi_G$ on $L(G)$. So $\widehat{\mathbb{G}}_a = (VN(G), \Gamma_G, \varphi_G, \psi_G)$ is a co-commutative LCQG.

Remark If G is a discrete group, then

$$\varphi_G(x) = \psi_G(x) = \langle x\delta_e | \delta_e \rangle.$$

Banach Algebra Structure on $L_1(\mathbb{G}) = L_\infty(\mathbb{G})_*$

The co-multiplication

$$\Gamma : L_\infty(\mathbb{G}) \to L_\infty(\mathbb{G}) \bar{\otimes} L_\infty(\mathbb{G})$$

induces an associative completely contractive multiplication

$$\star = \Gamma_* : f_1 \otimes f_2 \in L_1(\mathbb{G}) \hat{\otimes} L_1(\mathbb{G}) \to f_1 \star f_2 = (f_1 \otimes f_2) \circ \Gamma \in L_1(\mathbb{G})$$

on $L_1(\mathbb{G}) = M_*$ such that $A = (L_1(\mathbb{G}), \star)$ is a faithful completely contractive Banach algebra with

$$\langle L_1(\mathbb{G}) \star L_1(\mathbb{G}) \rangle = L_1(\mathbb{G}).$$

If $\mathbb{G}_a$ is a commutative LCQG, then $\star = \Gamma_{a*}$ is just the convolution on the convolution algebra

$$L_1(\mathbb{G}_a) = L_1(G).$$

If $\widehat{\mathbb{G}}_a$ is a co-commutative LCQG, then $\star = \widehat{\Gamma}_*$ is just the pointwise multiplication on the Fourier algebra

$$L_1(\widehat{\mathbb{G}}_a) = VN(G)_* = A(G).$$

Summary

For a locally compact quantum group $\mathbb{G}$, we have the following related algebras.

$$L_\infty(\mathbb{G}) = M$$

$$C_u(\mathbb{G}) \to C_0(\mathbb{G})$$

$$L_1(\mathbb{G})$$

$$M_u(\mathbb{G}) = C_u(\mathbb{G})^*$$

We say that $\mathbb{G}$ is discrete if $L_1(\mathbb{G})$ is unital, and $\mathbb{G}$ is compact if $C_0(\mathbb{G})$ is unital.

It is known that $\mathbb{G}$ is compact if and only if $\widehat{\mathbb{G}}$ is discrete.

When G is a Locally Compact Group

we have

$$L_\infty(\mathbb{G}) = L_\infty(G) \qquad L_\infty(\widehat{\mathbb{G}}) = VN(G)$$

$$C_u(\mathbb{G}) = C_0(\mathbb{G}) = C_0(G) \qquad C_r^*(G) \leftarrow C^*(G)$$

$$L_1(\mathbb{G}) = L_1(G) \qquad L_1(\widehat{\mathbb{G}}) = A(G)$$

$$M_u(\mathbb{G}) = M(G) \qquad M_u(\widehat{\mathbb{G}}) = B(G)$$

where

$$A(G) = \{f : G \to \mathbb{C} : f(s) = \langle \lambda_s \xi | \eta \rangle\} = VN(G)_*$$

is the Fourier algebra of G and $B(G) = C^*(G)^*$ be the Fourier-Stieltjes algebra of G.

Let $\mathbb{G} = (L_\infty(\mathbb{G}), \Gamma, \varphi, \psi)$ be a LCQG and let μ be a quantum measure, i.e. a states in $M_u(\mathbb{G}) = C_u^*(\mathbb{G})^*$. We can define a right multiplication map

$$m_\mu : f \in L_1(\mathbb{G}) \to f \star \mu \in L_1(\mathbb{G})$$

on $L_1(\mathbb{G})$ such that its adjoint map

$$\Phi_\mu = m_\mu^*$$

is a Markov operator on $L_\infty(\mathbb{G})$ and we can consider the space

$$\mathcal{H}_\mu = \{x \in L_\infty(\mathbb{G}) : \Phi_\mu(x) = x\}$$

of all μ-harmonic operators.

As we discussed before, there is a conditional expectation

$$\mathcal{E} : L_\infty(G) \to \mathcal{H}_\mu \subseteq L_\infty(G)$$

from $L_\infty(G)$ onto $\mathcal{H}_\mu$ given by the Banach limit

$$\langle \mathcal{E}(h), f \rangle = \lim_B \langle \Phi_\mu^n(h), f \rangle = \lim_B \langle h, f \star \mu^n \rangle$$

for all $h \in L_\infty(G)$ and $f \in L_1(G)$.

We can also consider $\mathcal{E}$ defined by Cesàro sums

$$\langle \mathcal{E}(h), f\rangle = \lim_{\mathcal{U}} \langle \frac{1}{n}(\Phi_\mu + \cdots + \Phi_\mu^n)(h), f\rangle = \lim_{\mathcal{U}} \langle h, \frac{1}{n} f \star (\mu + \cdots + \mu^n)\rangle$$

over any ultrafilder $\mathcal{U}$ on $\mathbb{N}$. We can obtain a von Neumann algebra product

$$x \circ y = \mathcal{E}(xy)$$

on $\mathcal{H}_\mu$, which is independent from the choice of $\mathcal{E}$.

We call the von Neumann algebra $(\mathcal{H}_\mu, \circ)$ the μ-Poisson boundary of $(\mathbb{G}, \mu)$.

Subalgebra Question

In this quantum group setting, we say that $\mu \in M_u(\mathbb{G})$ is non-degenerate if for any non-zero positive $x \in C_u(\mathbb{G})$, we have $\langle x, \mu^n \rangle \neq 0$ for some $n \in \mathbb{N}$.

Theorem [K-N-R]: Let $\mathbb{G}$ be a locally compact quantum group and μ a non-degerate state in $M_u(\mathbb{G})$. Then TFAE:

1) $\mathcal{H}_\mu$ is a subalgebra of $L_\infty(\mathbb{G})$;

2) $\mathcal{H}_\mu = \mathbb{C}1$.

Compact Quantum Group Case

Using a result of Franz and Skalski on the idempotent state on compact quantum groups, we can obtain the following result.

Theorem [K-N-R]: Let $\mathbb{G}$ be a compact quantum group and let μ be a state in $M_u(\mathbb{G}) = C_u(\mathbb{G})^*$. Then the Poisson boundary $\mathcal{H}_\mu$ is always a von Neumann subalgebra of $L_\infty(\mathbb{G})$!

If μ is non-degenerate, we must have $\mathcal{H}_\mu = \mathbb{C}1$.

A Characterization of Amenability

Theorem [K-N-R]: Let $\mathbb{G}$ be a locally compact quantum group with separable $L_1(\mathbb{G})$. Then TFAE:

1) $\mathbb{G}$ is amenable, i.e. there is a state $\phi : L_\infty(\mathbb{G}) \to \mathbb{C}$ such that

$$(\iota \otimes \phi) \circ \Gamma(x) = \phi(x)1$$

for all $x \in L_\infty(\mathbb{G})$.

2) There exists a quantum probability measure $\mu \in M_u(\mathbb{G}) = C_u(\mathbb{G})^*$ such that $\mathcal{H}_\mu = \mathbb{C}1$.

Some More Properties of Φ_μ

The Markov operator Φ_μ satisifes the following properties:

1. Φ_μ is fathful and invariant w.r.t the right Haar weight ψ, i.e.

$$\psi \circ \Phi_\mu = \psi$$

on $L_\infty(\mathbb{G})$.

2. Φ_μ satisfies the covariance condition

$$\Gamma \circ \Phi_\mu = (\iota \otimes \Phi_\mu) \circ \Gamma$$

3. It follows from 2) that

$$\alpha = \Gamma_{|\mathcal{H}_\mu} : \mathcal{H}_\mu \to L_\infty(\mathbb{G}) \bar{\otimes}_{\mathcal{F}} \mathcal{H}_\mu$$

defines a left coaction of $\mathbb{G}$ on the Poisson boundary $(\mathcal{H}_\mu, \circ)$, i.e α is a weak* continuous unital *-homomorphism injection such that

$$(\iota \otimes \alpha) \circ \alpha = (\Gamma \otimes \iota) \circ \alpha$$

Outline of Proof:

First, let us show that $\alpha = \Gamma_{|\mathcal{H}_\mu} : \mathcal{H}_\mu \to L_\infty(\mathbb{G})\bar{\otimes}_{\mathcal{F}}\mathcal{H}_\mu$.

Given $h \in \mathcal{H}_\mu \subseteq L_\infty(G)$, we have from 2) that

$$(\iota \otimes \Phi_\mu)\Gamma(h) = \Gamma(\Phi_\mu(h)) = \Gamma(h) \in L_\infty(G)\bar{\otimes}L_\infty(G).$$

Then for any $f \in L_\infty(G)$, the element

$$h_f = (f \otimes \iota)\Gamma(h) \in L_\infty(G)$$

is actually contained in $\mathcal{H}_\mu$ since

$$\Phi_\mu(h_f) = \Phi_\mu((f \otimes \iota)\Gamma(h)) = (f \otimes \iota)(\iota \otimes \Phi_\mu)\Gamma(h) = (f \otimes \iota)\Gamma(h) = h_f.$$

This shows that the element $h_f = (f \otimes \iota)\Gamma(h) \in \mathcal{H}_\mu$.

Therefore, Γ maps $\mathcal{H}_\mu$ into $L_\infty(G)\bar{\otimes}_{\mathcal{F}}\mathcal{H}_\mu$.

Next, we note that when we regard $\mathcal{H}_\mu$ as the von Neumann algebra, the above Fubini product coincides with the von Neumann algebra product, i.e. we have

$$L_\infty(\mathbb{G})\bar{\otimes}\mathcal{H}_\mu = L_\infty(\mathbb{G})\bar{\otimes}_{\mathcal{F}}\mathcal{H}_\mu.$$

We show that $\alpha = \Gamma_{|\mathcal{H}_\mu}$ is a unital *-homomorphism from von Neumann algebra $\mathcal{H}_\mu$ into the von Neumann algebra $L_\infty(G)\bar{\otimes}\mathcal{H}_\mu$.

Given $x, y \in \mathcal{H}_\mu$, we obtain that

$$\begin{aligned} \alpha(x \circ y) &= \Gamma(\mathcal{E}(xy)) = \Gamma(\lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^{n} \Phi_\mu^k(xy)) = \lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^{n} \Gamma(\Phi_\mu^k(xy)) \\ &= \lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^{n} (\iota \otimes \Phi_\mu^k)(\Gamma(xy)) = (\iota \otimes \mathcal{E})(\Gamma(x)\Gamma(y)) \\ &= \alpha(x) \circ \alpha(y) \in L_\infty(G)\bar{\otimes}\mathcal{H}_\mu. \end{aligned}$$

Since Γ is a comultiplication on $L_\infty(\mathbb{G})$, it is clear that α satisfies

$$(\iota \otimes \alpha) \circ \alpha = (\Gamma \otimes \iota) \circ \alpha.$$

So α defines a left coaction of $\mathbb{G}$ on $\mathcal{H}_\mu$.

Hence

$$\alpha = \Gamma_{|\mathcal{H}_\mu} : \mathcal{H}_\mu \to L_\infty(\mathbb{G}) \bar{\otimes} \mathcal{H}_\mu$$

defines a left co-action of $\mathbb{G}$ on $\mathcal{H}_\mu$.

Hence we can consider the crossed product

$$\mathcal{H}_\mu \rtimes \mathbb{G} = \{\Gamma_\mu(\mathcal{H}_\mu) \cup (L_\infty(\hat{\mathbb{G}}) \otimes 1)\}''.$$

What can we say about this crossed product ?

Extension to $B(L_2(\mathbb{G}))$

Theorem [J-N-R]: Every weak* continuous ucp map Φ on $L_\infty(\mathbb{G})$ satisfying covariance condition 2) has a unique weak* continuous ucp extension $\Theta(\mu)$ to $B(L_2(\mathbb{G}))$ such that

$$\Theta(\mu)(\hat{x}y\hat{z}) = \hat{x}\Theta(\mu(y)\hat{z}.$$

We can consider the Poisson boundary $\mathcal{H}_{\Theta(\mu)}$ of $(B(L_2(\mathbb{G})), \Theta(\mu))$ and it is seen that

$$\mathcal{H}_\mu \cup L_\infty(\hat{\mathbb{G}}) \subseteq \mathcal{H}_{\Theta(\mu)}.$$

Theorem [K-N-R]: We have the *-isomorphism

$$\mathcal{H}_\mu \rtimes \mathbb{G} = \mathcal{H}_{\Theta(\mu)}.$$

Therefore, $\mathcal{H}_\mu \rtimes \mathbb{G}$ is an injective von Neumann algebra !

This shows that the coaction $\mathbb{G}$ on $\mathcal{H}_\mu$ is amenable !

Remark: This result was first proved by Izumi (in 2004) for discrete groups. It was proved later on by Jawoski and Neufang (in 2007) for $L_\infty(G)$ case and by Neufang and Runde for $VN(G)$ case (requiring G has the AP).

Summary

$$\mu - \textbf{harmonic on } \mathbf{L_\infty(G)} \qquad \varphi - \textbf{harmonic on } \mathbf{VN(G)}$$

Chu and Lau

$$\mu - \textbf{harmonic on } \mathbf{B(L_2(G))}$$

Izumi, Jaworski and Neufang

$$\varphi - \textbf{harmonic on } \mathbf{B(L_2(G))}$$

Neufang and Runde

$$\mathcal{H}_{\Theta(\mu)} = \mathcal{H}_\mu \rtimes G \qquad \mathcal{H}_{\Theta(\varphi)} = \mathcal{H}_\varphi \rtimes \widehat{G}$$

undersomemildconditions

In general, we have

$$\mathcal{H}_{\Theta(\mu)} = \mathcal{H}_\mu \rtimes \mathbb{G}$$

for general locally compact quantum groups.

Thank you for your attention !