MARTINGALE HARDY SPACES

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1. Atomic decompostions for martingales, $0 < p \leq 1$;
2. Noncommutative Davis' decompostions, $0 < p \leq 1$.

ATOMIC DECOMPOSITIONS

- Harmonic Analysis: Coifman (1974) “Real characterizations of H_p-spaces”;
- Coifman and Weiss (1977)-survey- “Extensions of Hardy spaces and their use in analysis”.
- Martingale Theory: Herz (1974) “H_p-spaces of martingales, $0 < p \leq 1$ ”.
- Weisz (1990) “Martingale Hardy spaces for $0 < p \leq 1$”.
- Weisz (1994) Monograph:“Martingale Hardy spaces and their applications in Fourier analysis”.

Definition (Atoms)

Given a probability space $(\Omega, \mathcal{F}, P)$ and an increasing filtration $(\mathcal{F}_n)_{n\geq 1}$ of σ-subalgebras of $\mathcal{F}$, a function $f \in L_2(\Omega, \mathcal{F}, P)$ is called an *atom* if there exists $n \in \mathbb{N}$ and $A \in \mathcal{F}_n$ such that

- $\mathbb{E}_n(f) = 0$;
- $f = f\chi_A$;
- $\|f\|_2 \leq P(A)^{-1/2}$.

Definition (Atomic decompositions)

$$f = \sum_k \lambda_k f_k$$

where f_k's are atoms, $\lambda_k \in \mathbb{R}$ with $\sum_k |\lambda_k| < \infty$.

Noncommutative martingales

$\mathcal{M}$ is a finite von Neumann algebra with a normal faithful tracial state τ. For $0 < p \leq \infty$, denote by $L_p(\mathcal{M}, \tau)$ the associated non-commutative L_p-space i.e the completion of $\mathcal{M}$ under the norm

$$\|x\|_p := \tau\big((x^*x)^{p/2}\big)^{1/p}.$$

Let $(\mathcal{M}_n)_{n\geq 1}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of the $\mathcal{M}_n$'s is weak* dense in $\mathcal{M}$.

- For every $n \geq 1$, there is a normal trace preserving conditional expectation $\mathbb{E}_n : \mathcal{M} \to \mathcal{M}_n$.
- For every $1 \leq p < \infty$ and $n \geq 1$, $\mathbb{E}_n$ extends to a positive contraction

$$\mathbb{E}_n : L_p(\mathcal{M}, \tau) \to L_p(\mathcal{M}_n, \tau|_{\mathcal{M}_n}).$$

Definition

A non-commutative martingale with respect to the filtration $(\mathcal{M}_n)_{n\geq 1}$ is a sequence $x = (x_n)_{n\geq 1}$ in $L_1(\mathcal{M}, \tau)$ such that

$$\mathbb{E}_n(x_{n+1}) = x_n \quad \text{for all} \quad 1 \leq n < \infty.$$

If additionally $x \subset L_p(\mathcal{M}, \tau)$ for some $1 \leq p \leq \infty$, then x is called a L_p*-martingale*. In this case, we set

$$\|x\|_p := \sup_{n\geq 1} \|x_n\|_p \,.$$

If $\|x\|_p < \infty$, then x is called a L_p*-bounded martingale*.
The martingale difference sequence $dx = (dx_k)_{k\geq 1}$ associated to x is defined by

$$dx_k = x_k - x_{k-1}.$$

Square functions and Hardy spaces

$$S_{c,n}(x) = \Big(\sum_{k=1}^{n} |dx_k|^2\Big)^{1/2}, \quad S_c(x) = \Big(\sum_{k=1}^{\infty} |dx_k|^2\Big)^{1/2};$$

and

$$S_{r,n}(x) = \Big(\sum_{k=1}^{n} |dx_k^*|^2\Big)^{1/2}, \quad S_r(x) = \Big(\sum_{k=1}^{\infty} |dx_k^*|^2\Big)^{1/2}.$$

For $0 < p < \infty$. The column Hardy space $\mathcal{H}_p^c(\mathcal{M})$ is the completion of all finite L_∞-martingales under the norm $\|x\|_{\mathcal{H}_p^c} = \|S_c(x)\|_p$
The Hardy space for $0 < p < 2$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M})$$

equipped with the (quasi) norm

$$\|x\|_{\mathcal{H}_p} = \inf\big\{\|y\|_{\mathcal{H}_p^c} + \|z\|_{\mathcal{H}_p^r}\big\},$$

where the infimum is taken over all $y \in \mathcal{H}_p^c(\mathcal{M})$ and $z \in \mathcal{H}_p^r(\mathcal{M})$ such that $x = y + z$.

Conditioned version:

Let $x = (x_n)_{n\geq 1}$ be a finite martingale in $L_2(\mathcal{M})$. We set

$$s_{c,n}(x) = \Big(\sum_{k=1}^{n} \mathbb{E}_{k-1}|dx_k|^2\Big)^{1/2}, \quad s_c(x) = \Big(\sum_{k=1}^{\infty} \mathbb{E}_{k-1}|dx_k|^2\Big)^{1/2};$$

and

$$s_{r,n}(x) = \Big(\sum_{k=1}^{n} \mathbb{E}_{k-1}|dx_k^*|^2\Big)^{1/2}, \quad s_r(x) = \Big(\sum_{k=1}^{\infty} \mathbb{E}_{k-1}|dx_k^*|^2\Big)^{1/2}.$$

Let $0 < p < \infty$. Define $\mathsf{h}_p^c(\mathcal{M})$ and $\mathsf{h}_p^r(\mathcal{M})$) same way as before but using conditionned square functions.

We also need $\ell_p(L_p(\mathcal{M}))$, the space of all sequences $a = (a_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ such that

$$\|a\|_{\ell_p(L_p(\mathcal{M}))} = \Big(\sum_{n\geq 1} \|a_n\|_p^p\Big)^{1/p} < \infty \quad \text{if } 0 < p < \infty.$$

Let $\mathsf{h}_p^d(\mathcal{M})$ be the subspace of $\ell_p(L_p(\mathcal{M}))$ consisting of all martingale difference sequences.

We define the conditioned version of martingale Hardy spaces as follows: If $0 < p < 2$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^d(\mathcal{M}) + \mathsf{h}_p^c(\mathcal{M}) + \mathsf{h}_p^r(\mathcal{M})$$

equipped with the (quasi) norm

$$\|x\|_{\mathsf{h}_p} = \inf\big\{\|w\|_{\mathsf{h}_p^d} + \|y\|_{\mathsf{h}_p^c} + \|z\|_{\mathsf{h}_p^r}\big\},$$

where the infimum is taken over all $w \in \mathsf{h}_p^d(\mathcal{M}), y \in \mathsf{h}_p^c(\mathcal{M})$ and $z \in \mathsf{h}_p^r(\mathcal{M})$ such that $x = w + y + z$.

Noncommutative atoms

T. Bekjan, Z. Chen, M. Perrin, and Z. Yin (2010), "Atomic decomposition and interpolation for Hardy spaces", (JFA).

Definition

Let $0 < p \leq 1$. An operator $a \in L_2(\mathcal{M})$ is said to be a $(p,2)_c$-atom with respect to $(\mathcal{M}_n)_{n\geq 1}$, if there exist $n \geq 1$ and a projection $e \in \mathcal{M}_n$ such that:

(i) $\mathbb{E}_n(a) = 0$;

(ii) $r(a) \leq e$; $(ae = a)$;

(iii) $\|a\|_2 \leq \tau(e)^{1/2-1/p}$.

Replacing (ii) by (ii)′ $l(a) \leq e$, we have the notion of $(p,2)_r$-atoms.

Clearly, $(p,2)_c$-atoms and $(p,2)_r$-atoms are noncommutative analogues of $(p,2)$-atoms for classical martingales.

Definition (Column atomic Hardy space)

$\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M})$ is the space of all $x \in L_p(\mathcal{M})$ which admit a decomposition

$$x = \sum_k \lambda_k a_k$$

where for each k, a_k a $(p,2)_c$-atom or an element in $L_p(\mathcal{M}_1)$ of norm ≤ 1, and $\lambda_k \in \mathbb{C}$ satisfying $\sum_k |\lambda_k|^p < \infty$. We equip this space with the (quasi) norm

$$\|x\|_{\mathsf{h}_p^{c,\mathrm{at}}} = \inf \Big(\sum_k |\lambda_k|^p \Big)^{1/p}$$

where the infimum is taken over all decompositions of x.

$\mathsf{h}_p^{r,\mathrm{at}}(\mathcal{M})$ is define the same way;

$$\mathsf{h}_p^{\mathrm{at}}(\mathcal{M}) = \mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M}) + h_p^{r,\mathrm{at}}(\mathcal{M}) + h_p^d(\mathcal{M}).$$

MOTIVATION:

Theorem (Bekjan-Chen-Perrin-Yin (2010))

$$\mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M}) = \mathsf{h}_1^c(\mathcal{M}) \quad \textit{with equivalent norms.}$$

$$\|x\|_{\mathsf{h}_1^c} \leq \|x\|_{h_1^{c,\mathrm{at}}} \leq \sqrt{2}\|x\|_{\mathsf{h}_1^c}.$$

Question 1.

Constructive proof?

Question 2.

The case $0 < p \leq 1$?

Proposition

Let $0 < p \leq 1$. *If a is a* $(p,2)_c$*-atom then* $\|a\|_{\mathsf{h}_p^c} \leq 1$. *i.e.*

$$\mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M}) \subset \mathsf{h}_p^c(\mathcal{M}).$$

Theorem (R.-Xu)

For $0 < p \leq 1$,

$$\mathsf{h}_p^c(\mathcal{M}) = \mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M}) \quad \textit{with equivalent } p\textit{-norms};$$

$$\|x\|_{\mathsf{h}_p^c} \leq \|x\|_{\mathsf{h}_p^{c,at}} \leq C_p \|x\|_{\mathsf{h}_p^c}.$$

Corollary

For $0 < p \leq 1$,

$$\mathsf{h}_p(\mathcal{M}) = \mathsf{h}_p^{\mathrm{at}}(\mathcal{M}) \quad \textit{with equivalent } p\textit{-norms}.$$

For $\beta \geq 0$ define the Lipschitz space of order β by

$$\Lambda^c_\beta(\mathcal{M}) = \left\{ x \in L_2(\mathcal{M}) \;:\; \|x\|_{\Lambda^c_\beta} < \infty \right\},$$

where

$$\|x\|_{\Lambda^c_\beta} = \max\left\{ \|x_1\|_\infty,\; \sup_{n \geq 1} \sup_{e \in \mathcal{M}_n, \text{projection}} \frac{\|(x - x_n)e\|_2}{\tau(e)^{\beta + 1/2}} \right\}.$$

Corollary

Let $0 < p \leq 1$ *and* $\beta = \frac{1}{p} - 1$. *Then*

$$\left(\mathrm{h}^c_p(\mathcal{M})\right)^* = \Lambda^c_\beta(\mathcal{M}) \quad \textit{with equivalent norms.}$$

The construction: $2/3 < p \leq 1$.

- Fix a finite martingale $(x_n)_{1\leq n\leq N}$ in $L_2(\mathcal{M})$. Let $\alpha = 1 - (p/2)$. If $a \in L_p(\mathcal{M})$ is selfadjoint and $a = \int_{-\infty}^{\infty} s de_s^x$ is its spectral decomposition, then for any Borel subset $B \subseteq \mathbb{R}$, $\chi_B(x)$ denotes the corresponding spectral projection $\int_{-\infty}^{\infty} \chi_B(s) de_s^x$.
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$$\begin{cases} e_{k,0} &= \mathbf{1} \\ e_{k,n} &= \chi_{[0,2^{\alpha k}]}(e_{k,n-1} s_{c,n}^{\alpha}(x) e_{k,n-1}).e_{k,n-1} \text{ for } n \geq 1. \end{cases}$$
- For every $n \geq 1$, $e_{k,n} \in \mathcal{M}_{n-1}$.
$$\begin{cases} p_{0,n} &:= \bigwedge_{j=0}^{\infty} e_{j,n} \\ p_{k,n} &:= \bigwedge_{j=k}^{\infty} e_{j,n} - \bigwedge_{j=k-1}^{\infty} e_{j,n} \quad \text{for } k \geq 1. \end{cases} \tag{1}$$
- Then for every $n \geq 1$, $\sum_k p_{k,n} = \mathbf{1}$.

- (First step) Split (x_n) into two martingales $(x_n^{(0)})$ and $(x_n^{(1)})$ with

$$dx_n^{(0)} = \sum_{k=0}^{\infty} dx_n s_n^{-\alpha} p_{k,n} s_n^{\alpha} (\bigwedge_{j \geq k} e_{j,n})$$

and

$$dx_n^{(1)} = \sum_{k=0}^{\infty} dx_n s_n^{-\alpha} (\bigwedge_{j \geq k-1} e_{j,n}) s_n^{\alpha} p_{k,n}$$

$x = x^{(0)} + x^{(1)}$.

Lemma

$$\sum_n \| dx_n s_n^{-\alpha} \|_2^2 \leq \frac{2}{p} \| x \|_{h_p^c}^p .$$

Lemma

$$\sum_k 2^{pk} \tau(\mathbf{1} - \bigwedge_{j \geq k} e_{j,N}) \leq C_p \| x \|_{h_p^c}^p , \quad \alpha < p.$$

For general $2/(2\nu+1) < p \leq 2/(2\nu-1)$, one needs to split x into 2^ν-martingales.
[the case $2/5 < p \leq 2/3$]

$$dx_n^{(0)} = \sum_{k=0}^{\infty} dx_n s_n^{-\alpha} p_{k,n} s_n^{\alpha/2} (\bigwedge_{j\geq k} e_{j,n}) s_n^{\alpha/2} (\bigwedge_{j\geq k} e_{j,n})$$

$$dx_n^{(1)} = \sum_{m=1}^{\infty} (\sum_{k=0}^{m-1} dx_n s_n^{-\alpha} p_{k,n} s_n^{\alpha/2} (\bigwedge_{j\geq k} e_{j,n}) s_n^{\alpha/2}) p_{m,n}$$

$$dx_n^{(2)} = \sum_{k=0}^{\infty} dx_n s_n^{-\alpha} (\bigwedge_{j\geq k-1} e_{j,n}) s_n^{\alpha/2} p_{k,n} s_n^{\alpha/2} (\bigwedge_{j\geq k} e_{j,n})$$

$$dx_n^{(3)} = \sum_{m=1}^{\infty} (\sum_{k=0}^{m-1} dx_n s_n^{-\alpha} (\bigwedge_{j\geq k-1} e_{j,n}) s_n^{\alpha/2} p_{k,n} s_n^{\alpha/2}) p_{m,n}$$

Sketch of the decompostion for $x^{(1)}$.($2/3 < p \leq 1$)

$$\begin{cases} q_{0,n} & := \bigwedge\limits_{j\geq 0} e_{j,n} \\ q_{k,n} & := \bigwedge\limits_{j\geq k-1} e_{j,n-1} - \bigwedge\limits_{j\geq k-1} e_{j,n} \quad \text{for } k \geq 1, \end{cases} \tag{2}$$

Then for any given $n \geq 1$ and $k \geq 1$, we have that $\sum_{l=1}^{n} q_{k,l} = \mathbf{1} - \bigwedge_{j\geq k-1} e_{j,n}$. In particular, $p_{k,n} \leq \sum_{l=1}^{n} q_{k,l}$.

$$\begin{aligned} x^{(1)} &= \sum_{k,n} dx_n s_n^{-\alpha} (\bigwedge_{j\geq k-1} e_{j,n}) s_n^{\alpha} p_{k,n} \\ &= \sum_{k,n} dx_n s_n^{-\alpha} (\bigwedge_{j\geq k-1} e_{j,n}) s_n^{\alpha} p_{k,n} (\sum_{l=1}^{n} q_{k,l}) \\ &= \sum_{k,l} \Big(\sum_{n\geq l} dx_n s_n^{-\alpha} (\bigwedge_{j\geq k-1} e_{j,n}) s_n^{\alpha} p_{k,n} q_{k,l} \Big) \end{aligned}$$

$$x^{(1)} = \sum_{k,l} \Big(\sum_{n \geq l} dx_n s_n^{-\alpha} \big(\bigwedge_{j \geq k-1} e_{j,n} \big) s_n^{\alpha} p_{k,n} q_{k,l} \Big) = \sum_{k,l} b_{k,l}$$

- $\mathbb{E}_l(b_{l,k}) = 0$, $r(b_{k,l}) \leq q_{k,l}$ with $q_{k,l} \in \mathcal{M}_l$.
- $a_{k,l} = \frac{b_{k,l}}{\|b_{k,l}\|_2 \tau(q_{k,l})^{1/p-1/2}}$ is an atom
- $\lambda_{k,l} = \|b_{k,l}\|_2 \tau(q_{k,l})^{1/p-1/2}$
- $x^{(1)} = \sum_{k,l} \lambda_{k,l} a_{k,l}$.
- Need to prove $\sum_{k,l} \lambda_{k,l}^p < \infty$.

- $I = \sum_{l,k} 2^{pk} \tau(q_{k,l}) = \sum_k 2^{pk} \tau(\mathbf{1} - \bigwedge_{j \geq k-1} e_{j,N}) \leq 2^p C_p \|x\|_{\mathsf{h}_p^c}^p$.
- $$\begin{aligned} II = \sum_{k,l} 2^{-2\alpha k} \|b_{k,l}\|_2^2 &= \sum_{k,l} 2^{-2\alpha k} \| \sum_{n \geq l} dx_n s_n^{-\alpha} (\bigwedge_{j \geq k-1} e_{j,n}) s_n^{\alpha} p_{k,n} q_{k,l} \|_2^2 \\ &= \sum_{n,k} 2^{-2\alpha k} \sum_{l \leq n} \|dx_n s_n^{-\alpha} (\bigwedge_{j \geq k-1} e_{j,n}) s_n^{\alpha} p_{k,n} q_{k,l} \|_2^2 \\ &= \sum_{n,k} 2^{-2\alpha k} \|dx_n s_n^{-\alpha} (\bigwedge_{j \geq k-1} e_{j,n}) s_n^{\alpha} p_{k,n} \|_2^2 \\ &\leq \sum_n \|dx_n s_n^{-\alpha}\|_2^2 \leq c_p \|x\|_{\mathsf{h}_p^c}^p. \end{aligned}$$
- $\sum_{k,l} \lambda_{k,l}^p \leq I^{\alpha}.II^{1-\alpha} \leq C_p \|x\|_{\mathsf{h}_p^c}^p$.

$$x^{(0)} = \sum_n \sum_{k=0}^{\infty} dx_n s_n^{-\alpha} p_{k,n} s_n^{\alpha} (\bigwedge_{j \geq k} e_{j,n}).$$

$$\Gamma_{k,n} := p_{k,n} s_n^{\alpha} (\bigwedge_{j \geq k} e_{j,n})$$

$$E_{k,n} := support(\sum_{l \leq n} \sum_{m \geq k} |\Gamma_{m,l}|^2)$$

$$\begin{aligned} x^{(0)} &= \sum_n \sum_{k=0}^{\infty} dx_n s_n^{-\alpha} p_{k,n} s_n^{\alpha} (\bigwedge_{j \geq k} e_{j,n}) E_{k,n} \\ &= \sum_n \sum_{k=0}^{\infty} dx_n s_n^{-\alpha} p_{k,n} s_n^{\alpha} (\bigwedge_{j \geq k} e_{j,n}) (\sum_{l \leq n} E_{k,l} - E_{k,l-1}) \\ &= \sum_{l,k} (\sum_{n \geq l} dx_n s_n^{-\alpha} p_{k,n} s_n^{\alpha} (\bigwedge_{j \geq k} e_{j,n}) (E_{k,l} - E_{k,l-1}). \end{aligned}$$

$$x^{(0)} = \sum_{l,k} (\sum_{n \geq l} dx_n s_n^{-\alpha} p_{k,n} s_n^{\alpha} (\bigwedge_{j \geq k} e_{j,n})(E_{k,l} - E_{k,l-1}) = \sum_{k,l} \widetilde{b}_{k,l}.$$

- $\mathbb{E}_l(\widetilde{b}_{k,l}) = 0, \quad r(\widetilde{b}_{k,l}) \leq E_{k,l} - E_{k,l-1} = \widetilde{q}_{k,l} \in \mathcal{M}_l,$
- $\widetilde{a}_{k,l} = \frac{\widetilde{b}_{k,l}}{\|\widetilde{b}_{k,l}\|_2 \tau(\widetilde{q}_{k,l})^{1/p-1/2}}$ is an atom;
- $\widetilde{\lambda}_{k,l} = \|\widetilde{b}_{k,l}\|_2 \tau(\widetilde{q}_{k,l})^{1/p-1/2}$
- $x^{(0)} = \sum_{k,l} \widetilde{\lambda}_{k,l} \widetilde{a}_{k,l}$.
- $\sum_{k,l} \widetilde{\lambda}_{k,l}^p \leq c_p \|x\|_{\mathsf{h}_p^c}^p$.

The argument extents to $1 < p < 2$ (with the definition of atoms fo this range is the same as before)

$$\mathsf{h}_p^c(\mathcal{M}) \subset \mathsf{h}_p^{c,\mathrm{at}}(\mathcal{M}).$$

Davis type decompositions

Theorem (Davis, 1970)

If $f \in \mathcal{H}_1(\Omega, \Sigma, \mathbb{P})$ *then* $\exists$ *a decomposition* $f = g + h$ *with*

$$\|s(g)\|_1 + \sum_n \|dh_n\|_1 \leq C\|f\|_{\mathcal{H}_1}$$

i.e., $\mathcal{H}_1 = \mathsf{h}_1$ *with equivalent norms.*

Theorem (Perrin, 2009)

$\mathcal{H}_1(\mathcal{M}) = \mathsf{h}_1(\mathcal{M})$ *with equivalent norms. More precisely, if* $x \in \mathcal{H}_1(\mathcal{M})$,

$$\frac{1}{2}\|x\|_{\mathsf{h}_1} \leq \|x\|_{\mathcal{H}_1} \leq \sqrt{6}\|x\|_{\mathsf{h}_1}.$$

Junge's conditionned L_p-spaces

For a finite sequence $a = (a_n)_{n\geq 1}$, we define (with the convention that $\mathbb{E}_0 = \mathbb{E}_1$)

$$\|a\|_{L_p^{cond}(\mathcal{M},\ell_2^c)} = \Big\| \Big(\sum_{n\geq 1} \mathbb{E}_{n-1}(a_n^* a_n) \Big)^{1/2} \Big\|_p .$$

- $L_p^{cond}(\mathcal{M}, \ell_2^c)$ is the completion of the space of finites sequences in $\mathcal{M}$ with the quasi-norm $\| \cdot \|_{L_p^{cond}(\mathcal{M},\ell_2^c)}$
- $L_p^{cond}(\mathcal{M}, \ell_2^r)$ is defined by $a \in L_p^{cond}(\mathcal{M}, \ell_2^r)$ if and only if $a^* \in L_p^{cond}(\mathcal{M}, \ell_2^c)$ with $\|a\|_{L_p^{cond}(\mathcal{M},\ell_2^r)} = \|a^*\|_{L_p^{cond}(\mathcal{M},\ell_2^c)}$.
- $\mathrm{h}_p^c(\mathcal{M})$ and $\mathrm{h}_p^r(\mathcal{M})$ are subspaces of $L_p^{cond}(\mathcal{M}, \ell_2^c)$ and $L_p^{cond}(\mathcal{M}, \ell_2^r)$ respectively.

Theorem (R.-Xu)

Let $0 < p \leq 1$. Then for every bounded L_2 martingale x there exist two adapted sequences a^c and a^d with the following properties:

(i) $dx_n = a_n^c + a_n^d, \forall\, n \geq 1, ;$

(ii) $\big\|a^c\big\|_{L_p^{\mathrm{cond}}(\mathcal{M};\ell_2^c)} + \big\|a^d\big\|_{\ell_p(L_p(\mathcal{M}))} \leq C_p\big\|x\big\|_{\mathcal{H}_p^c};$

(iii) $\big\|a^c\big\|_{L_2^{\mathrm{cond}}(\mathcal{M};\ell_2^c)} + \big\|a^d\big\|_{\ell_2(L_2(\mathcal{M}))} \leq C_p\big\|x\big\|_{\mathcal{H}_2^c}.$

Corollary

Every bounded L_2 martingale x admits a decomposition into two martingales x^c and x^d such that

(i) $x = x^c + x^d;$

(ii) $\big\|x^c\big\|_{\mathsf{h}_1^c} + \big\|x^d\big\|_{\mathsf{h}_1^d} \leq C\big\|x\big\|_{\mathcal{H}_1^c};$

(iii) $\big\|x^c\big\|_{\mathsf{h}_2^c} + \big\|x^d\big\|_{\mathsf{h}_2^d} \leq C\big\|x\big\|_{\mathcal{H}_2^c}.$

Consequently, $\mathcal{H}_1^c(\mathcal{M}) = \mathsf{h}_1^c(\mathcal{M}) + \mathsf{h}_1^d(\mathcal{M})$ with equivalent norms.

The case $2/3 < p \leq 1$

- Fix a finite martingale $(x_n)_{1\leq n\leq N}$ in $L_2(\mathcal{M})$. Let $\alpha = 1 - (p/2)$.
- $$\begin{cases} e_{k,0} &= \mathbf{1} \\ e_{k,n} &= \chi_{[0,2^{\alpha k}]}(e_{k,n-1,}S^{\alpha}_{c,n}(x)e_{k,n-1}).e_{k,n-1} \ \text{ for } n \geq 1. \end{cases} \tag{3}$$
- For every $n \geq 1$, $e_{k,n} \in \mathcal{M}_n$.

$$\begin{cases} p_{0,n} &:= \bigwedge_{j=0}^{\infty} e_{j,n} \\ p_{k,n} &:= \bigwedge_{j=k}^{\infty} e_{j,n} - \bigwedge_{j=k-1}^{\infty} e_{j,n} \quad \text{for } k \geq 1. \end{cases} \tag{4}$$

- Then for every $n \geq 1$, $\sum_k p_{k,n} = \mathbf{1}$.

(First step) Split (x_n) into two adapted sequences a and b with

$$a_n = \sum_{k=0}^{\infty} dx_n S_n^{-\alpha} p_{n,k} S_n^{\alpha} (\bigwedge_{j \geq k} e_{j,n-1})$$

and

$$b_n = \sum_{k=0}^{\infty} dx_n S_n^{-\alpha} (\bigwedge_{j \geq k-1} e_{j,n}) S_n^{\alpha} p_{k,n-1}$$

Lemma

$$\|b\|_{L_p^{cond}(l_2^c)} \leq c_p \|x\|_{\mathcal{H}_p^c}.$$

$$a_n = \sum_{k=0}^{\infty} dx_n S_n^{-\alpha} p_{n,k} S_n^{\alpha}(\bigwedge_{j\geq k} e_{j,n-1})$$

$$\Gamma_{k,n} = p_{n,k} S_n^{\alpha}(\bigwedge_{j\geq k} e_{j,n-1})$$

$$E_{k,n} = s(\sum_{m\geq k} \sum_{l\leq n} |\Gamma_{m,l}|^2) \in \mathcal{M}_n$$

Define

$$a_n^{(1)} = \sum_{k=0}^{\infty} dx_n S_n^{-\alpha} p_{n,k} S_n^{\alpha}(\bigwedge_{j\geq k} e_{j,n-1}) E_{k,n-1}$$

and

$$a_n^{(2)} = \sum_{k=0}^{\infty} dx_n S_n^{-\alpha} p_{n,k} S_n^{\alpha}(\bigwedge_{j\geq k} e_{j,n-1})(E_{k,n} - E_{k,n-1})$$

Then $a = a^{(1)} + a^{(2)}$.

Lemma

$$\|a^{(1)}\|_{L_p^{cond}(l_2^c)} \leq c_p \|x\|_{\mathcal{H}_p^c}.$$

Lemma

$$\|a^{(2)}\|_{\ell_p(L_p(\mathcal{M}))} \leq c_p \|x\|_{\mathcal{H}_p}$$

and

$$\|a^{(2)}\|_{\ell_2(L_2(\mathcal{M}))} \leq c_p \|x\|_{\mathcal{H}_2}$$

Take

$$a^c = b + a^{(1)} \quad \text{and} \quad a^d = a^{(2)}.$$

1. Since $\mathcal{H}_1^c(\mathcal{M}) = \mathsf{h}_1^c(\mathcal{M})$, we have $\mathcal{H}_1^c(\mathcal{M}) = \mathsf{h}_1^{c,\mathrm{at}}(\mathcal{M})$ i.e., every martingale in $\mathcal{H}_1^c(\mathcal{M})$ admits atomic decompositions.
2. For $0 < p < 1$, elements of $\mathcal{H}_p^c(\mathcal{M})$ do not necessary admit atomic decompositions. In fact, even for the commutative case,

$$\mathcal{H}_p^c(\mathcal{M}) \not\subseteq \mathsf{h}_p^c(\mathcal{M}).$$

3. q-atoms (Hong and Mei-2011).