Operator space structures on L_p-spaces

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Plan

The non-commutative Khintchine inequalities are mainly due to Françoise Lust-Piquard [LP1986] (see also [LPP1992])
They play a very important rôle in the recent developments in Operator Space Theory and non-commutative L_p-spaces, gateway to many other topics unconditional convergence

• Martingale inequalities in non-commutative L_p-spaces (cf. P.-Xu, Randriantoanina, Randriantoanina-Parcet, Junge-Xu, Junge-Le Merdy-Xu).

• Connection with Grothendieck's Theorem :
The operator space analogues of Grothendieck's Theorem also close connection with some form of Khintchine inequalities (cf. Junge-P. GAFA 1994, P. -Shlyakhtenko, Invent. 2002, Haagerup-Musat 2008, Regev-Vidick 2012).

Further motivation in
Random Matrix Theory and
Free Probability.
The Rademacher functions in classical L_p (i.e. i.i.d. ± 1-valued independent random variables) satisfy the same inequalities as the freely independent ones in non-commutative L_p for $p < \infty$.

Let (ε_k) be **classical random signs on** (Ω, μ) **:**

$$\varepsilon_k = \pm 1$$

$$\mu\{\varepsilon_k = \pm 1\} = 1/2$$

(realized by the Rademacher functions on $\Omega = [0, 1]$).

Classical Khintchine inequalities :

$\forall 0 < p < \infty \; \exists A_p > 0$ and $\exists B_p > 0$ such that $\forall x = (x_n) \in \ell_2$

$$A_p \left(\sum |x_n|^2 \right)^{1/2} \leq \left(\int \left| \sum x_n \varepsilon_n \right|^p d\mu(\varepsilon) \right)^{1/p} \leq B_p \left(\sum |x_n|^2 \right)^{1/2}$$

Note $A_p = 1$ if $p \geq 2$ and $B_p = 1$ if $p \leq 2$.

[Françoise Lust-Piquard,1986] proved an important non-commutative analogue, BUT

Cases $p < 2$ and $p > 2$ are different ! !

Operator spaces

Definition

(“Non-commutative Banach spaces") An operator space is a subspace of $B(\mathcal{H})$, i.e. we are given

$$E \subset B(H)$$

“Operator space Theory" is now well developed after Ruan’s1987 thesis cf. Effros-Ruan, Blecher-Paulsen...

Non-commutative norm

Definition

(“Non-commutative norm") Given

$$E \subset B(H)$$

$$M_n(E) \subset B(H \oplus \cdots \oplus H) = B(\ell_2^n \otimes H)$$

and more generally if $B = B(\ell_2)$

$$B \otimes_{\min} E \subset B(\ell_2 \otimes H)$$

Minimal Tensor Product

Given

$$E \subset B(H) \qquad F \subset B(\mathcal{H})$$

we define

$$E \otimes_{\min} F \subset B(H \otimes_2 \mathcal{H})$$

("spatial" or "minimal" tensor product)

$$B = B(H)$$

Quantization : Scalars replaced by operators
Banach space structure on E : a norm on $\mathbb{C} \otimes E$:
$(E \simeq \mathbb{C} \otimes E)$

$$\left\| \sum c_k \otimes x_k \right\| \quad c_k \in \mathbb{C}\ x_k \in E$$

Operator space structure on E : a norm on $B \otimes E$

$$\left\| \sum b_k \otimes x_k \right\| \quad b_k \in B\ x_k \in E$$

RUAN's Fundamental Theorem gives a correspondence :
<Norm on $B \otimes E$> $\leftrightarrow$ <o.s.s. on E>
cf. Effros-Ruan, Blecher-Paulsen...
o.s.s.=operator space structure

Completely bounded (C.B.) maps

E, F operator spaces
Definition : A linear map $u: \ E \to F$ is c.b. if

$$Id_B \otimes u : B \otimes_{\min} E \to B \otimes_{\min} F$$

is bounded and

$$\|u\|_{cb} = \|Id_B \otimes u\|$$

$u: \ E \to F$ is completely isometric if $Id_B \otimes u$ is isometric.
We denote $CB(E, F)$ associated Banach space
By Ruan's theorem **it "becomes" an operator space** with "non-com. norm" defined by

$$B \otimes_{\min} CB(E, F) \stackrel{def}{=} CB(E, B \otimes_{\min} F)$$

Operations on Operator Spaces

Defined using Ruan's Theorem :
-duality : $E^* = CB(E, \mathbb{C})$
-quotient spaces
-complex interpolation
Easy :
-complex conjugation assuming $E \subset B(\ell_2)$

$$[a_{ij}] \in E \Leftrightarrow [\overline{a_{ij}}] \in \bar{E}$$

more generally

$$E \subset B(H) \qquad \bar{E} \subset B(\bar{H}) \qquad (\ell_2 \simeq \bar{\ell}_2)$$

“The" Operator Hilbert space OH

Given a Hilbert space H there is a UNIQUE o.s.s. on H such that with this o.s.s.

$$H = \overline{H^*} \quad \text{completely isometrically}$$

We denote it by

$$OH$$

Note :
$H \simeq K$ (isometrically) IFF $OH \simeq OK$ (completely isometrically)

Example 1 $E = O\ell_2$ $\quad (e_n)$ o.n. basis of ℓ_2 $\quad b_n \in B$
$S = \sum b_n \otimes e_n \in B \otimes E$

$$\|S\|_{(2)} = \|\sum b_n \otimes \bar{b}_n\|^{1/2}_{B(H \otimes_2 \bar{H})}$$

Example 2 $E = OL_2(\Omega, \mu)$ $\quad f \in B \otimes L_2(\Omega, \mu)$ $\quad f : \Omega \to B$

$$\|f\|_{(2)} = \|\int f(\omega) \otimes \overline{f(\omega)} d\mu(\omega)\|^{1/2}_{B(H \otimes_2 \bar{H})} = \|\int f \dot{\otimes} \bar{f} \, d\mu\|^{1/2}_{B(H \otimes_2 \bar{H})}$$

where

$$(f \dot{\otimes} \bar{f})(\omega) \overset{def}{=} f(\omega) \otimes \overline{f(\omega)} \in B \otimes \bar{B}$$

RECALL : Classical Khintchine inequalities :

$$\varepsilon_k = \pm 1$$

$$\mu\{\varepsilon_k = \pm 1\} = 1/2$$

$\forall 0 < p < \infty \quad \forall x = (x_n) \in \ell_2$

$$A_p \left(\sum |x_n|^2\right)^{1/2} \leq \left(\int \left|\sum x_n \varepsilon_n\right|^p d\mu(\varepsilon)\right)^{1/p} \leq B_p \left(\sum |x_n|^2\right)^{1/2}$$

Obvious $A_p = 1$ if $p \geq 2$ and $B_p = 1$ if $p \leq 2$.

Operator space interpretation of Khintchine In. :

Using complex interpolation
L_p-spaces (commutative or not) become operator spaces
As Banach spaces

$$\overline{\text{span}}\{\varepsilon_n\} \simeq \ell_2$$

but as operator spaces, Lust-Piquard 1986 implies

$$\overline{\text{span}}\{\varepsilon_n\} \simeq R_p \cap C_p \quad \text{if} \quad 2 \le p < \infty$$

and

$$\overline{\text{span}}\{\varepsilon_n\} \simeq R_p + C_p \quad \text{if} \quad 1 \le p \le 2$$

As Banach spaces

$$\overline{\mathrm{span}}\{\varepsilon_n\} \simeq \ell_2$$

Preceding results hold for the “natural" operator space structure on L_p defined using complex interpolation
However, the goal of this talk is to show that there an alternative operator space structure on L_p for which we have as operator space

$$\overline{\mathrm{span}}\{\varepsilon_n\} \simeq O\ell_2 \quad \forall \quad 1 \leq p < \infty$$

Recall OH is uniquely characterized completely isometrically by

$$OH \simeq \overline{OH^*}$$

Another o.s.s. on L_p

$L_p = L_p(\Omega, \mathcal{A}, \mu)$
We identify an element $f \in B(H) \otimes L_p$ with a function $f: \Omega \to B(H)$
Given n measurable functions $f_j: \Omega \to B(H_j)$, we denote by $f_1 \dot{\otimes} \cdots \dot{\otimes} f_n$ the pointwise product viewed as a function with values in $B(H_1) \otimes \cdots \otimes B(H_n) \subset B(H_1 \otimes_2 \cdots \otimes_2 H_n)$.
Then if $p = 2m$ we set

$$\|f\|_{(p)} = \left\| \int f \dot{\otimes} \bar{f} \dot{\otimes} \cdots \dot{\otimes} f \dot{\otimes} \bar{f} \right\|^{1/p}$$

where $f \dot{\otimes} \bar{f}$ is repeated $p/2$-times and the norm is that of $B(H \otimes_2 \bar{H} \otimes_2 \cdots \otimes_2 H \otimes_2 \bar{H})$.

Theorem

For any $p = 2m$ this defines an o.s.s. on L_p. We denote by Λ_p the corresponding o.s.

When $p = 2$, we have $L_2 \simeq \ell_2$ isometrically and Λ_2 is the same as $O\ell_2$ but otherwise we only have a completely contractive inclusion

$$L_p \subset \Lambda_p$$

The key result is a version of Hölder ineq. : ($m = 1$ due to Haagerup)

Lemma

$m \in \mathbb{N}$, $p = 2m \geq 2$ $\forall f_1, \ldots, f_p \in B(H) \otimes L_p$ we have

$$\left\| \int f_1 \dot{\otimes} \cdots \dot{\otimes} f_p \, d\mu \right\| \leq \prod_{k=1}^{p} \|f_k\|_{(p)} \tag{1}$$

$$\|f\|_{(p)} = \sup\{ \left\| \int f \dot{\otimes} f_2 \dot{\otimes} \cdots \dot{\otimes} f_p \, d\mu \right\| \mid \sup_{j \geq 2} \|f_j\|_{(p)} \leq 1 \}$$

Return to

$$Rad_p \subset L_p$$

Theorem

For any $p = 2m$ the span of the functions $\{\varepsilon_n\}$ is completely isomorphic to $O\ell_2$, i.e. there is a constant C_p such that for any $f \in B \otimes span\{\varepsilon_n\}$, say $f = \sum b_n \otimes \varepsilon_n$, we have

$$(1/C_p)\|\sum b_n \otimes \bar{b}_n\|^{1/2}_{B(H \otimes_2 \bar{H})} \leq \|f\|_{(p)} \leq C_p\|\sum b_n \otimes \bar{b}_n\|^{1/2}_{B(H \otimes_2 \bar{H})}$$

Moreover, the orthogonal projection onto $\mathrm{span}\{\varepsilon_n\}$ is c.b. on Λ_p.

Recall

$$\|(b_n)\|_{[OH]} = \|\sum b_n \otimes \bar{b}_n\|^{1/2}_{B(H \otimes_2 \bar{H})}$$

Another non-commutative Burkholder inequality

More generally consider $f \in B \otimes L_p$ and assume we have a filtration $(\mathcal{A}_n)$, let $f_n = \mathbb{E}^{\mathcal{A}_n} f$, and let $d_0 = f_0$ and $d_n = f_n - f_{n-1}$ for all $n \geq 1$. For any $p = 2^k$ there is a constant C'_p such that

$$(1/C'_p) \| \sum d_n \dot{\otimes} \bar{d}_n \|^{1/2}_{(p/2)} \leq \|f\|_{(p)} \leq C'_p \| \sum d_n \dot{\otimes} \bar{d}_n \|^{1/2}_{(p/2)}$$

Square function :

$$\sum d_n \dot{\otimes} \bar{d}_n$$

Moreover :
The Hilbert transform is completely bounded on $\Lambda_p(\mathbb{T})$.

non-commutative L_p

(M,τ) semi-finite non-com. proba. space

$$f = \sum b_j \otimes x_j \in B \otimes M \qquad f^* = \sum \bar{b}_j \otimes x_j^* \in \bar{B} \otimes M$$

Lemma

Let $p \geq 2$ be an even integer. Consider $f_j \in B(H_j) \otimes L_p(\tau)$. Let

$$\|f_j\|_{(p)} = \|\hat{\tau}(f_j^* \dot{\otimes} f_j \dot{\otimes} \cdots \dot{\otimes} f_j^* \dot{\otimes} f_j)\|^{1/p}_{B(\overline{H_j} \otimes H_j \otimes \cdots \otimes \overline{H_j} \otimes H_j)}$$

where $f_j^* \dot{\otimes} f_j$ is repeated $p/2$ times. We have then

$$\|\hat{\tau}(f_1 \dot{\otimes} \cdots \dot{\otimes} f_p)\| \leq \prod_{j=1}^{p} \|f_j\|_{(p)},$$

$$\textit{also} \qquad \|f\|_{(p)} = \|f^*\|_{(p)}$$

Theorem

Let $p \geq 2$ be an even integer. The space $L_p(\tau)$ can be equipped with an o.s.s. so that denoting by $\Lambda_p(\tau)$ the resulting operator space we have for any H and any f in $B(H) \otimes L_p(\tau)$

$$\|f\|_{B(H)\otimes_{\min}\Lambda_p(\tau)} = \|f\|_{(p)}.$$

Buchholz (2001 + Bull. PAN 2005) Let $p = 2m$. $\{x_k\} \subset L_p(\tau)$, has p-th moments defined by pairing-function $F\colon\ P_2(2m) \to \mathbb{C}$ defined on the set if 2-partitions of $\{1, \ldots, 2m\}$ if

$$\forall k_1, \ldots, k_{2m} \quad \tau(x_{k_1} x_{k_2}^* x_{k_3} \ldots x_{k_{2n-1}} x_{k_{2m}}^*) = \sum_{\nu \sim (k_1, \ldots, k_{2m})} F(\nu)$$

$\nu \sim (k_1, \ldots, k_{2m})$ means $k_i = k_j$ whenever $\{i, j\}$ is a block of ν.

Theorem

Let $p = 2m$. Let $\{x_k\} \subset L_p(\tau)$, with p-th moments defined by pairing-function $F\colon\ P_2(2m) \to \mathbb{C}$.
Then $\forall f = \sum x_j \otimes b_j$ ($b_j \in B(H)$), we have

$$\left\|\sum b_k \otimes \bar{b}_k\right\|^{1/2} \leq \|f\|_{(p)} \leq C \left\|\sum b_k \otimes \bar{b}_k\right\|^{1/2},$$

where $C = \left(\sum_{\nu \in P_2(2n)} |F(\nu)|\right)^{1/2m}$.

A surprise

When $p \to \infty$ $\quad f \in B \otimes L_\infty(\mu) \quad \mu(\Omega) = 1$

$$\|f\|_{(p)} \to \|f\|_{B \otimes_{\min} L_\infty(\mu)} = \|f\|_{L_\infty(\mu;B)}$$

roughly

$$\Lambda_p(\mu) \to L_\infty(\mu)$$

However, in non-commutative case no longer true

When $p \to \infty$ $\quad f \in B \otimes L_\infty(\tau) = B \otimes \mathcal{M} \quad \tau(1) = 1$

$$\|f\|_{(p)} \to \|f\|_{B \otimes_{\min} CB(OL_2(\tau))}$$

So we find roughly

$$\Lambda_p(\tau) \to \underline{\mathcal{M}} \subset CB(OL_2(\tau)).$$

In general if $E \subset B(H)$ let

$$\underline{E} \subset CB(OH).$$

For any $u : \ E \to F$ we have

$$\|\underline{u} : \ \underline{E} \to \underline{F}\|_{cb} \leq \|u : \ E \to F\|_{cb}$$

$$u \textit{ complete isometry} \Rightarrow \underline{u} \textit{ complete isometry}$$

Consequence : The o.s.s. of $\underline{E}$ depends only on that of E

New version of Haagerup's free Khintchine Ineq.

Let G be a free group with $n \geq 2$ generators, $\mathcal{M}$ its von Neumann algebra let W_d denote the words of length d in G Then for any finitely supported function $b: \ W_d \to B$ we have

$$\left\|\sum_{t \in W_d} b(t) \otimes \bar{b}(t)\right\|^{1/2} \leq \|\sum_{t \in W_d} b(t) \otimes \lambda(t)\|_{B \otimes \underline{\mathcal{M}}} \leq$$

$$(d+1)\left\|\sum_{t \in W_d} b(t) \otimes \bar{b}(t)\right\|^{1/2}.$$

Thus in $\underline{\mathcal{M}}$ and in $\Lambda_p(\tau_G)$ $(\forall p \in 2\mathbb{N})$

$$\overline{\operatorname{span}}\{\lambda(t) \mid t \in W_d\} \simeq O\ell_2(W_d)$$

Thank you !