Hypercontractivity of the Ornstein-Uhlenbeck and Poisson semigroups for free products

Mathilde Perrin

Instituto de Ciencias Matemáticas
Madrid

—Joint work with Marius Junge, Carlos Palazuelos, Javier Parcet and Éric Ricard—

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A. Fourier Analysis

For $f \in L_p(\mathbb{T})$ we consider the Fourier coefficients $\widehat{f}(k) = \int_{\mathbb{T}} f(x) e^{-2\pi i k x} \, dx$

- Riemann-Lebesgue: $f \in L_1(\mathbb{T}) \Rightarrow \widehat{f}(k) \to 0$ (slow convergence).
- Plancherel: $f \in L_2(\mathbb{T}) \Rightarrow \sum_{k \in \mathbb{Z}} |\widehat{f}(k)|^2 < \infty$ (faster convergence).
- Hausdorff-Young: $f \in L_p(\mathbb{T})$ $(1 < p < 2) \Rightarrow \sum_k k^{2\alpha} |\widehat{f}(k)|^2 < \infty$ $(\alpha < \frac{1}{2} - \frac{1}{p})$.

Classical Problem Find conditions for $\psi : \mathbb{Z} \to \mathbb{C}$ such that

$$f \in L_p(\mathbb{T}) \Rightarrow \sum_k |\psi(k)|^2 |\widehat{f}(k)|^2 < \infty.$$

i) $\psi = \chi_\Omega \to \Lambda_p$-sets (Rudin, Bourgain, Talagrand...).

ii) ψ rational $\to$ Hausdorff-Young inequality given above.

iii) ψ exponential $\to$ If $\psi(k) = r^{|k|} = e^{-t|k|} \to$ Poisson semigroup $e^{-t|\cdot|}$.

Hypercontractivity of the Poisson semigroup. For $1 < p \leq q < \infty$ we have

$$\Big\| \sum_k e^{-t|k|} \widehat{f}(k) \exp_k \Big\|_q \leq \|f\|_p \ \Leftrightarrow \ t \geq \frac{1}{2} \log \Big(\frac{q-1}{p-1} \Big).$$

Bonami '70 + Beckner '75 + Weissler '80.

B. Quantum field theory

The problem of the existence of quantum fields lead (in the late 60's) to look for lower bounds for Hamiltonians of the form $H = A + V$, where A is an elliptic second order differential operator, and V is a potential. Assume that A is a Dirichlet form on $\mathbb{R}^n$, i.e., there is a measure μ_A such that

$$\langle Af, g\rangle = \int_{\mathbb{R}^n} \nabla f \cdot \nabla g \; d\mu_A.$$

Important examples:

- Lebesgue measure $\rightsquigarrow A_m f(x) = -\Delta f(x)$
- Gaussian measure $d\mu_\gamma(x) = \dfrac{\exp(-|x|^2/2)}{(2\pi)^{\frac{n}{2}}} dx \rightsquigarrow A_\gamma f(x) = -\Delta f(x) + x \cdot \nabla f(x)$

Hypercontractivity of the Ornstein-Uhlenbeck semigroup

$$(\mathrm{HC}_{\mu_\gamma}) \qquad \left\| e^{-tA_\gamma} f \right\|_q \leq \|f\|_p \;\Leftrightarrow\; t \geq \frac{1}{2} \log\left(\frac{q-1}{p-1}\right) \quad \text{for} \quad 1 < p \leq q < \infty.$$

Nelson '66 + Glimm '68 + Segal '70 + Nelson '73 + $\cdots$

Corollary [Nelson '73]. For V real and $H_\gamma = A_\gamma + V$ we have

$$\big\langle H_\gamma f, f \big\rangle_{L_2(\mu_\gamma)} \geq -\|f\|^2_{L_2(\mu_\gamma)} \log \|e^{-V}\|_{L_2(\mu_\gamma)} \quad (\text{dim } \leq \infty).$$

C. Logarithmic Sobolev inequalities

$$(\mathrm{LS}_\mu) \qquad \int_{\mathbb{R}^n} |f(x)|^2 \log |f(x)|^2 d\mu(x) \leq 2 \int_{\mathbb{R}^n} |\nabla f(x)|^2 d\mu(x) + \|f\|^2_{L_2(\mu)} \log \|f\|^2_{L_2(\mu)}.$$

Gross '75 + Weissler '78 + Carlen '91 + Beckner '92 + $\cdots$

Remarks:

- Dimension independent, infinitesimal version of Sobolev inequalities:

  $$\|f\|_{L_q(\mathbb{R}^n,dx)} \leq C_{p,n} \||\nabla f|\|_{L_p(\mathbb{R}^n,dx)} \quad \text{for } f \in C_c^\infty,\ \frac{1}{q} = \frac{1}{p} - \frac{1}{n},\ 1 \leq p < \infty.$$

- $(\mathrm{LS}_\mu) \Leftrightarrow (\mathrm{HC}_\mu)$ where $\langle A_\mu f, g\rangle = \int_{\mathbb{R}^n} \nabla f \cdot \nabla g \; d\mu.$

Some known results of HC and Applications

- HC for $\mathbb{Z}_2 \rightsquigarrow$ Bonami-Beckner 2-point inequality

$$\Big(\frac{1}{2}\Big[(\frac{1+r}{2})a + (\frac{1-r}{2})b\Big]^q + \frac{1}{2}\Big[(\frac{1-r}{2})a + (\frac{1+r}{2})b\Big]^q \Big)^{\frac{1}{q}} \leq \Big(\frac{a^p + b^p}{2}\Big)^{\frac{1}{p}}.$$

Some known results of HC and Applications

- HC for $\mathbb{Z}_2 \rightsquigarrow$ Bonami-Beckner 2-point inequality
- HC for Clifford algebras $\rightsquigarrow$ Gaussians Fermions

$$\left\|e^{-tA_{-1}}f\right\|_q \leq \|f\|_p \;\Leftrightarrow\; t \geq \frac{1}{2}\log\left(\frac{q-1}{p-1}\right) \quad [A_{-1} = \text{Femionic number operator}].$$

Gross '75 $+$ Ball-Carlen-Lieb '94 $+$ Carlen-Lieb '93.

Some known results of HC and Applications

- HC for $\mathbb{Z}_2 \rightsquigarrow$ Bonami-Beckner 2-point inequality
- HC for Clifford algebras $\rightsquigarrow$ Gaussians Fermions
- HC for q-deformed algebras $\rightsquigarrow$ Free Gaussians, Araki-Woods factors...

  a) Type II $\longrightarrow$ Biane '97.

  b) Type III $\longrightarrow$ Lee-Ricard '11.

Some known results of HC and Applications

- HC for $\mathbb{Z}_2 \rightsquigarrow$ Bonami-Beckner 2-point inequality
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- HC for complex manifolds $\rightsquigarrow$ shorter contraction time for holomorphic functions

$$\left\|e^{-tA_{\pm 1}} f\right\|_q \leq \|f\|_p \;\Leftrightarrow\; t \geq \frac{1}{2}\log\left(\frac{q}{p}\right).$$

Janson '83 + Gross '99 + Kemp '05 + Krolac '10.

Some known results of HC and Applications

- HC for $\mathbb{Z}_2 \rightsquigarrow$ Bonami-Beckner 2-point inequality
- HC for Clifford algebras $\rightsquigarrow$ Gaussians Fermions
- HC for q-deformed algebras $\rightsquigarrow$ Free Gaussians, Araki-Woods factors...
- HC for complex manifolds $\rightsquigarrow$ shorter contraction time for holomorphic functions
- Connections / Applications

  i) Differential Geometry
  Ricci Curvature (Bakry)
  Poincaré Conjecture (Perelman)

  ii) Information Theory
  Entropy inequalities (Faris)
  Bell inequalities (Regev-de Wolf)

  iii) Other related results
  Concentration of measures (Herbst, Ledoux)
  Optimal Hausdorff-Young inequalities (Beckner)...

- Survey

L. Gross, Hypercontractivity, logarithmic Sobolev inequalities and applications: A survey of surveys. Diffusion, quantum theory, and radically elementary mathematics: a celebration of Edward Nelson's contribution to science. Ed. by William G. Faris. Princeton University Press, 2006.

Aim

Study hypercontractivity for free products in the two following cases:

- ⇝ Ornstein-Uhlenbeck semigroup
- ⇝ Poisson semigroup

Definition

We say that a semigroup $(P_t)_{t\geq 0}$ is Hypercontractive with constant α if for $1 < p \leq q < \infty$ we have

$$\|P_t\|_{L_p \to L_q} \leq 1 \quad \text{for} \quad t \geq \frac{\alpha}{2} \log\left(\frac{q-1}{p-1}\right).$$

If $\alpha = 1$ we say that the semigroup is Hypercontractive with optimal constant.

Remark

Note that by Gross' argument, which has been extended to the noncommutative L_p spaces by Olkiewicz and Zegarlinski ('98), this will give Logarithmic Sobolev inequalities of the form

$$\tau(|f|^2 \log |f|^2) \leq 2\alpha\langle Nf, f\rangle + \|f\|_2^2 \log \|f\|_2^2, \quad \text{for} \quad P_t = e^{-tN}.$$

In the tracial case, this reduces the problem to the study of

Hypercontractivity $L_2 \to L_q$ for $q \geq 2$.

Spin Systems with mixed Commutation and Anti-commutation Relations

Let I be a finite subset of $\mathbb{Z}$ and $\varepsilon : I \times I \to \{-1, 1\}$ be such that

$$\varepsilon(i,j) = \varepsilon(j,i) \quad \text{and} \quad \varepsilon(i,i) = -1.$$

We consider the Spin Algebra $\mathcal{A}(I, \varepsilon)$ with generators $(x_i)_{i \in I}$ and relations

$$x_i x_j - \varepsilon(i,j) x_j x_i = 2\delta_{i,j} \quad \text{for } i, j \in I.$$

- Vector Basis: $x_A = x_{i_1} \cdots x_{i_p}$ if $A = \{i_1, \cdots, i_p\} \subset I$, and $x_\emptyset = 1$.
- Antilinear involution: $x_i^* = x_i$.
- Trace: $\tau(x_A) = \delta_{A,\emptyset}$.

$\rightsquigarrow$ von Neumann algebra structure.

Definition
The ε-Ornstein-Uhlenbeck semigroup on $\mathcal{A}(I, \varepsilon)$ is given by

$$P_t^\varepsilon x_A = e^{-t|A|} x_A \quad \text{for } A \subset I.$$

Theorem[Carlen-Lieb '92 + Biane '97]
The ε-Ornstein-Uhlenbeck semigroup P_t^ε is hypercontractive with optimal constant.

1.Free product and O-U semigroup (Probabilistic approach)

Theorem A
Let $(\mathcal{A}_\beta)_\beta$ be a family of Spin algebras, and denote by $(P_t^\beta)_\beta$ the corresponding family of Ornstein-Uhlenbeck semigroups. Then the free product $*_\beta P_t^\beta$ is hypercontractive with optimal constant on $*_\beta \mathcal{A}_\beta$.

Here $*_\beta P_t^\beta$ is given by

$$*_\beta P_t^\beta (x_{A_1}^{\beta_1} x_{A_2}^{\beta_2} \cdots x_{A_m}^{\beta_m}) = e^{-t(|A_1|+|A_2|+\cdots+|A_m|)} x_{A_1}^{\beta_1} x_{A_2}^{\beta_2} \cdots x_{A_m}^{\beta_m}$$

for $\beta_1 \neq \beta_2 \neq \cdots \neq \beta_m$ and $x_{A_i}^{\beta_i} \in \mathcal{A}_{\beta_i}$.

Corollary
With an extra use of Speicher's Central Limit Theorem, we obtain that the free product of Ornstein-Uhlenbeck semigroups is hypercontractive with optimal constant on the free product of q-deformed algebras $*\Gamma_q(\mathcal{H})$.

Observation
In particular, $\mathcal{L}(\mathbb{Z}_2)$ is a Spin algebra (with only one generator), and the Ornstein-Uhlenbeck semigroup coincide with the Poisson semigroup on $\mathcal{L}(\mathbb{Z}_2)$. Moreover, we have

$$\mathcal{L}(\mathbb{Z}_2) * \cdots * \mathcal{L}(\mathbb{Z}_2) = \mathcal{L}(\mathbb{Z}_2 * \cdots * \mathbb{Z}_2).$$

2. Bonami-Beckner Theorem for free product

Let G be a discrete group and $\mathcal{L}(G) \subset B(\ell_2(G))$ be the associated group von Neumann algebra.

The Bonami-Beckner Theorem extends to $\mathbb{Z}_2^n$ and $\mathbb{Z}^n$.
We consider the two following free products:

$$\rightsquigarrow G = \underbrace{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{n}$$

$$\rightsquigarrow G = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n} = \mathbb{F}_n$$

For these groups, we define the usual length function

$$|w| = \text{ number of letters in the reduced word}$$

and the associated Poisson semigroup

$$P_t(f) = \sum_{w \in G} e^{-t|w|} \hat{f}(w)\lambda(w), \quad \text{for } f = \sum_{w \in G} \hat{f}(w)\lambda(w) \in \mathcal{L}(G).$$

Remark

$P_t = e^{-tN}$ where N is given by $N(f) = \sum_{w \in G} |w| \hat{f}(w)\lambda(w)$.

2.a) Bonami-Beckner Theorem for free product (Probabilistic approach)

Poisson semigroup: $P_t(f) = \sum_{w \in G} e^{-t|w|} \hat{f}(w)\lambda(w)$ for $f = \sum_{w \in G} \hat{f}(w)\lambda(w) \in \mathcal{L}(G)$.

As a direct consequence of Theorem A we have

Corollary

1) $G = \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$:The Poisson semigroup P_t is hypercontractive with optimal constant.

2) $G = \mathbb{Z} * \cdots * \mathbb{Z} = \mathbb{F}_n$:The Poisson semigroup P_t is hypercontractive with constant 2.

Proof

$1) \Rightarrow 2)$ We use the embedding

$$\phi : \begin{cases} \mathbb{F}_n & \hookrightarrow & (\mathbb{Z}_2 * \mathbb{Z}_2)^{*n} \\ g_i & \mapsto & u_i v_i \end{cases},$$

where $g_1, \cdots, g_n$ denote the generators of $\mathbb{F}_n$ and u_i (resp. v_i) is the generator of the first (resp. second) copy of $\mathbb{Z}_2$ in the i-th copy of $\mathbb{Z}_2 * \mathbb{Z}_2$.

Observe that $|\phi(w)| = 2|w|$ for $w \in \mathbb{F}_n$.

2.a) Bonami-Beckner Theorem for free product (Probabilistic approach)

Poisson semigroup: $P_t(f) = \sum_{w \in G} e^{-t|w|} \hat{f}(w) \lambda(w)$ for $f = \sum_{w \in G} \hat{f}(w)\lambda(w) \in \mathcal{L}(G)$.

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Logarithmic Sobolev Inequalities

For any $f \in \mathcal{L}(G)$ we have

1) $G = \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$:

$$\tau(|f|^2 \log |f|^2) \leq 2 \sum_{w \in G} |w| \hat{f}(w)^2 + \|f\|_2^2 \log \|f\|_2^2.$$

2) $G = \mathbb{Z} * \cdots * \mathbb{Z} = \mathbb{F}_n$:

$$\tau(|f|^2 \log |f|^2) \leq 4 \sum_{w \in G} |w| \hat{f}(w)^2 + \|f\|_2^2 \log \|f\|_2^2.$$

2.b) Bonami-Beckner Theorem for free product (Combinatorial approach)

Theorem B

Let $G = \mathbb{F}_2$.

Then the Poisson semigroup P_t is hypercontractive with optimal constant from $L_2(\mathcal{L}(\mathbb{F}_2))$ to $L_q(\mathcal{L}(\mathbb{F}_2))$, for q large even integer ($q \in 2\mathbb{N}$ and $q \geq 7044$), i.e.,

$$\|P_t(f)\|_q \leq \|f\|_2 \quad \Leftrightarrow \quad t \geq \frac{1}{2}\log(q-1).$$

Theorem C

Let $G = \mathbb{F}_n$.

Then there exists a constant $\alpha(n) \geq 1$ such that the Poisson semigroup P_t is hypercontractive with constant $\alpha(n)$ from $L_2(\mathcal{L}(\mathbb{F}_n))$ to $L_4(\mathcal{L}(\mathbb{F}_n))$, i.e.,

$$\|P_t(f)\|_4 \leq \|f\|_2 \quad \text{for} \quad t \geq \frac{\alpha(n)}{2}\log(3).$$

Corollary

Let $G = \mathbb{F}_n$.

Then there exists a constant $\alpha(n) \geq 1$ such that the Poisson semigroup P_t is hypercontractive with constant $\alpha(n)\log(3)$, i.e., for $1 < p \leq q < \infty$ we have

$$\|P_t(f)\|_q \leq \|f\|_p \quad \text{for} \quad t \geq \frac{\alpha(n)\log(3)}{2}\log\left(\frac{q-1}{p-1}\right).$$

2.c) Bonami-Beckner Theorem for free product
(Comparison of the two approaches - $G = \mathbb{F}_2$)

Theorem B

Let $G = \mathbb{F}_2$.
Then the Poisson semigroup P_t is hypercontractive with optimal constant from $L_2(\mathcal{L}(\mathbb{F}_2))$ to $L_q(\mathcal{L}(\mathbb{F}_2))$, for q large even integer ($q \in 2\mathbb{N}$ and $q \geq 7044$), i.e.,

$$\|P_t(f)\|_q \leq \|f\|_2 \quad \Leftrightarrow \quad t \geq \frac{1}{2}\log(q-1).$$

Remarks

- With more computations we could improve the range $q \geq 7044$.
- Interpolation + Log Sobolev inequalities
  $\rightsquigarrow$ Hypercontractivity of the Poisson semigroup with constant $\alpha \simeq 4,43$
  $\rightsquigarrow$ the probabilistic approach gives a better constant for the HC $L_p \to L_q$.
- Interpolation
  $\rightsquigarrow$ the combinatorial approach gives a better constant for the HC $L_2 \to L_q$ for $q \geq 7044$.

2.c) Bonami-Beckner Theorem for free product
(Comparison of the two approaches - $G = \mathbb{F}_n$)

Corollary of Theorem C

Let $G = \mathbb{F}_n$.

Then there exists a constant $\alpha(n) \geq 1$ such that the Poisson semigroup P_t is hypercontractive with constant $\alpha(n)\log(3)$, i.e., for $1 < p \leq q < \infty$ we have

$$\|P_t(f)\|_q \leq \|f\|_p \quad \text{for} \quad t \geq \frac{\alpha(n)\log(3)}{2}\log\Big(\frac{q-1}{p-1}\Big).$$

Remarks

- $\alpha(n)\log(3) \sim \log(2n)$ for n large.
- We have

$$\alpha(2)\log(3) < 2 \quad \text{and} \quad \alpha(3)\log(3) < 2$$

$\rightsquigarrow$ the combinatorial approach gives a better constant for the HC for $\mathbb{F}_2$ and $\mathbb{F}_3$,
the probabilistic one gives a better constant for the HC for $\mathbb{F}_n$, $n \geq 4$.