Twisted Hilbert transforms and idempotent Fourier multipliers

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—based on joint works with M. Perrin, E. Ricard and K.M. Rogers—

Operator spaces, Quantum probability and Applications

Wuhan University

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Operator algebra methods in harmonic analysis and quantum information

Instituto de Ciencias Matemáticas – Madrid

From May 20th to July 17th, 2013

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Operator spaces, harmonic analysis and quantum probability

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Directional Hilbert transforms

- Hilbert transform. Given a Schwartz function $f : \mathbb{R} \to \mathbb{C}$

$$Hf(x) = \mathsf{p.v.} \int_{\mathbb{R}} \frac{f(s)}{x-s} ds \ \to \ \widehat{Hf}(\xi) = -i \, \mathsf{sgn}(\xi) \widehat{f}(\xi).$$

As the model for the so-called Calderón-Zygmund operators

- ◦ $H : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ for $1 < p < \infty$,
- ◦ $H : L_1(\mathbb{R}) \to L_{1,\infty}(\mathbb{R})$ and $H : L_\infty(\mathbb{R}) \to \mathrm{BMO}(\mathbb{R})$.

This yields convergence results for the Fourier transform on $\mathbb{T}$ and $\mathbb{R}$

$$\lim_{\mathrm{N}\to\infty} \Big\| f - \sum_{|n| \le \mathrm{N}} \widehat{f}(n) \exp(2\pi i n \, \cdot) \Big\|_{L_p(\mathbb{T})} = 0,$$

$$\lim_{\mathrm{R}\to\infty} \Big\| f - \int_{|\xi| \le \mathrm{R}} \widehat{f}(\xi) \exp(2\pi i \xi \, \cdot) d\xi \Big\|_{L_p(\mathbb{R})} = 0.$$

Directional Hilbert transforms

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- Riesz transforms [Kernel extension]. Given $u \in \mathrm{S}^{n-1}$, we set

$$R_u f(x) = \Big\langle \text{p.v.} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n+1}} f(y) \, dy, \; u \Big\rangle \;\to\; \widehat{R_u f}(\xi) = -i \frac{\langle \xi, u \rangle}{|\xi|} \widehat{f}(\xi).$$

This is again a CZO and satisfies the same L_p estimates and endpoint estimates.

Directional Hilbert transforms

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- Directional Hilbert transforms [Multiplier extension]. Given $u \in \mathrm{S}^{n-1}$

$$\widehat{H_u f}(\xi) = -i\ \text{sgn}\langle \xi, u\rangle\ \widehat{f}(\xi),$$

$$H_u f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(x-su)}{s}\, ds.$$

  - Multiplier singularity in $\mathbb{R}^n \ominus \mathbb{R}u \rightarrow$ Not CZO's.
  - L_p-bdness and weak L_1-bdness still ok by Fubini.
  - Convergence of Fourier inversion for convex polyhedra.
  - Fefferman th: $f \mapsto \int_{\mathrm{B}_1(0)} \widehat{f}(\xi) \exp(2\pi i \langle \xi, x\rangle)\, d\xi$ unbded on $L_p(\mathbb{R}^2)$, $p \neq 2$.

Cocycle form on discrete group vNa's

Given a discrete group G, a cocycle is any triple $\psi = (\mathcal{H}, \alpha, b)$ formed by a real Hilbert space $\mathcal{H}$, an orthogonal action $\alpha : \mathrm{G} \to \mathrm{Aut}_{\mathbb{R}}(\mathcal{H})$ and a map $b : \mathrm{G} \to \mathcal{H}$ satisfying the cocycle law

$$\alpha_g(b(h)) = b(gh) - b(g).$$

In the line of [JMP], we consider the maps

$$R_{\psi,u} : \sum_{g \in \mathrm{G}} \widehat{f}(g)\lambda(g) \ \mapsto \ -i \sum_{g \in \mathrm{G}} \frac{\langle b(g), u\rangle_{\mathcal{H}}}{\|b(g)\|_{\mathcal{H}}} \ \widehat{f}(g)\lambda(g),$$

$$H_{\psi,u} : \sum_{g \in \mathrm{G}} \widehat{f}(g)\lambda(g) \ \mapsto \ -i \sum_{g \in \mathrm{G}} \mathrm{sgn}\langle b(g), u\rangle_{\mathcal{H}} \, \widehat{f}(g)\lambda(g).$$

In general, take $T_{\psi,m} : \lambda(g) \mapsto m(b(g))\lambda(g)$ for any multiplier m in $\mathcal{H} \simeq \mathbb{R}^n$.

Problem. Is $R_{\psi,u}$ an L_p-bounded map for $1 < p < \infty$? What about $H_{\psi,u}$?

Theorem [Junge/Mei/P]. *If* $\dim \mathcal{H} < \infty$ *and* $1 < p < \infty$

a) $R_{\psi,u} : L_p(\mathcal{L}(\mathrm{G}), \tau) \to L_p(\mathcal{L}(\mathrm{G}), \tau)$ *for all unit vector* $u \in \mathcal{H}$.

b) $T_{\psi,m} : L_p(\mathcal{L}(\mathrm{G}), \tau) \to L_p(\mathcal{L}(\mathrm{G}), \tau)$ *for Hörmander-Mihlin type* m*'s*.

Difficulty for $H_{\psi,u}$: sgn$\langle \cdot, u\rangle_{\mathcal{H}}$ is singular away from 0, not smooth enough...

The 'donut multipliers' for $\mathrm{G} = \mathbb{R}$

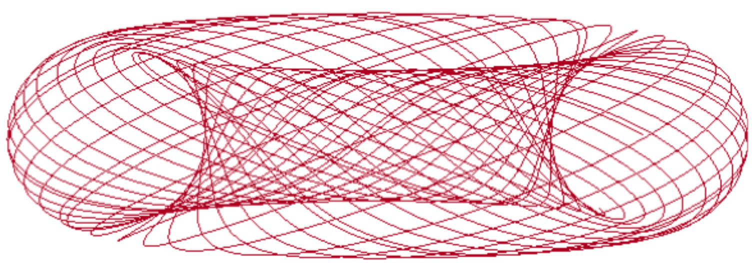
Given $f : \mathbb{R} \to \mathbb{C}$, consider the multiplier

$$\widehat{T_m f}(\xi) \ = \ \underbrace{m\big(\cos 2\pi\alpha\xi - 1, \sin 2\pi\alpha\xi, \cos 2\pi\beta\xi - 1, \sin 2\pi\beta\xi\big)}_{m(b(\xi))} \ \widehat{f}(\xi)$$

for any $\alpha, \beta \in \mathbb{R}$ and any $m \in \mathcal{C}^6(\mathbb{R}^4 \setminus \{0\})$ satisfying HM. Then, $T_m : L_p(\mathbb{R}) \to L_p(\mathbb{R})$.

$$\Downarrow$$

If we choose α/β irrational
$b(\xi)$ defines a geodesic flow on $\mathbb{T}^2$ with dense orbit



What happens if m is the characteristic function of a semispace / convex polyhedron?
These 'chaotic' idempotent multipliers are finite products of $H_{\psi,u}$'s

Twisted Hilbert transforms

- According to K. de Leeuw's theorem

$$H_u : \sum_{\xi \in \mathbb{R}^n} \widehat{f}(\xi) \textsf{b-exp}_\xi \mapsto -i \sum_{\xi \in \mathbb{R}^n} \textsf{sgn}\langle \xi, u \rangle \widehat{f}(\xi) \textsf{b-exp}_\xi$$

  defines a bounded map $L_p(\mathbb{R}^n_{\mathrm{bohr}}, \mu) \to L_p(\mathbb{R}^n_{\mathrm{bohr}}, \mu)$ for any $1 < p < \infty$.

- Given a cocycle $\psi = (\mathcal{H}, \alpha, b)$ with $\dim \mathcal{H} = n$

$$\lambda(g) \in \mathcal{L}(\mathrm{G}) \mapsto \textsf{b-exp}_{b(g)} \rtimes_\alpha \lambda(g) \in \mathcal{L}(\mathbb{R}^n_{\mathrm{disc}}) \rtimes_\alpha \mathrm{G}$$

  defines a trace preserving $*$-homomorphism. In particular, we find that

$$H_u \rtimes_\alpha id_{\mathrm{G}} \quad L_p\big(\mathcal{L}(\mathbb{R}^n_{\mathrm{disc}}) \rtimes_\alpha \mathrm{G}\big)\text{-bounded} \ \Rightarrow \ H_{\psi,u} \quad L_p(\mathcal{L}(\mathrm{G}))\text{-bounded}.$$

- Note that $H_u \rtimes_\alpha id_{\mathrm{G}} = H_{\phi,u}$ for certain (simple) cocycle $\phi = (\mathcal{K}, \beta, d)$ on Γ_{disc}.

- Given any orthogonal representation $\gamma : \mathrm{G} \to O(n)$
  - Are twisted Hilbert transforms $H_u \rtimes_\gamma id_{\mathrm{G}}$ L_p-bounded?
  - Can we replace $\Gamma_{\mathrm{disc}} = \mathbb{R}^n_{\mathrm{disc}} \rtimes_\gamma \mathrm{G}$ by the group $\Gamma = \mathbb{R}^n \rtimes_\gamma \mathrm{G}$?

A characterization for twisted Hilbert transforms

Theorem [P/Rogers]. *If* $1 < p \neq 2 < \infty$, *tfae*

i) *The map* $H_u \rtimes_\gamma id_{\mathrm{G}}$ *is bounded on* $L_p(\mathcal{L}(\Gamma))$,

ii) *The map* $H_u \rtimes_\gamma id_{\mathrm{G}}$ *is bounded on* $L_p(\mathcal{L}(\Gamma_{\mathrm{disc}}))$,

iii) *The* γ*-orbit of* u $\mathcal{O}_\gamma(u) = \big\{ \gamma_g(u) \, | \, g \in \mathrm{G} \big\}$ *is finite*,

iv) *The following matrix inequality holds*

$$\int_{\mathbb{R}^n} \Big\| \Big(H_{\gamma_{g^{-1}}(u)}(f_{g,h})(x) \Big) \Big\|_{S_p(\mathrm{G})}^p \, dx \, \leq c_p \int_{\mathbb{R}^n} \Big\| \Big(f_{g,h}(x) \Big) \Big\|_{S_p(\mathrm{G})}^p \, dx.$$

We may also prove $L_1 \to L_{1,\infty}$ *and* $L_\infty \to \mathrm{BMO}$ *type estimates for finite orbits.*

Remarks.

- Condition iii) in sharp contrast with behavior of $R_u \rtimes_\gamma id_{\mathrm{G}}$!
- Strategy i) $\Rightarrow$ ii) $\Rightarrow$ iii) $\Rightarrow$ i) and as a consequence iii) $\Leftrightarrow$ iv).
- Equivalence i) $\Leftrightarrow$ ii) is a twisted form of K. de Leeuw's theorem.
- Neuwirth/Ricard transference $\longrightarrow$ Weaker condition than iv) for G amenable.
- In particular, the operators $H_{\psi,u}$'s are in general NOT L_p-bounded $\longrightarrow$ When?

Sketch of the proof I: Meyer's square function inequality

We may assume (wlog) $p > 2$

$$H_u \rtimes_\gamma id_{\mathrm{G}} : L_p(\mathcal{L}(\Gamma_{\mathrm{disc}})) \longrightarrow L_p(\mathcal{L}(\Gamma_{\mathrm{disc}}))$$

($\gamma_g H_u \gamma_g^{-1} = H_{\gamma_g(u)}$ + NC Littlewood-Paley [JMP])

$$\Downarrow$$

$$\Big\| \Big[\sum_{j=1}^\infty |H_{\gamma_{g_j}(u)}(f_{g_j})|^2 \Big]^{\frac12} \Big\|_{L_p(\mathbb{R}^n_{\mathrm{bohr}})} \lesssim \Big\| \Big[\sum_{j=1}^\infty |f_{g_j}|^2 \Big]^{\frac12} \Big\|_{L_p(\mathbb{R}^n_{\mathrm{bohr}})} + \Big\| \Big[\sum_{j=1}^\infty |\gamma_{g_j^{-1}} f_{g_j}|^2 \Big]^{\frac12} \Big\|_{L_p(\mathbb{R}^n_{\mathrm{bohr}})}$$

(de Leeuw's decompactification + L_p-norm for almost periodic functions)

$$\Downarrow$$

$$\Big\| \Big[\sum_{j=1}^\infty |H_{\gamma_{g_j}(u)}(f_{g_j})|^2 \Big]^{\frac12} \Big\|_{L_p(\mathbb{R}^n)} \lesssim \Big\| \Big[\sum_{j=1}^\infty |f_{g_j}|^2 \Big]^{\frac12} \Big\|_{L_p(\mathbb{R}^n)} + \lim_{\mathrm{M}\to\infty} \mathrm{M}^{\frac{n}{p}} \Big\| \Big[\sum_{j=1}^\infty |\pi_{\mathrm{M}}^{g_j} \gamma_{g_j^{-1}} f_{g_j}|^2 \Big]^{\frac12} \Big\|_p$$

(Ergodic theory + Suitable choice of f_g's)

$$\Downarrow$$

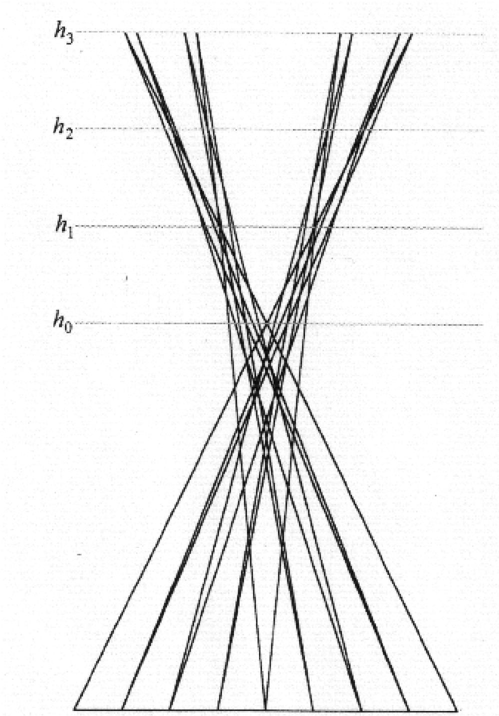
$$\underbrace{\Big\| \Big[\sum_{j=1}^\infty |H_{\gamma_{g_j}(u)}(f_{g_j})|^2 \Big]^{\frac12} \Big\|_{L_p(\mathbb{R}^n)} \lesssim \Big\| \Big[\sum_{j=1}^\infty |f_{g_j}|^2 \Big]^{\frac12} \Big\|_{L_p(\mathbb{R}^n)}}_{\text{Meyer's inequality}} \quad \text{for} \quad f_{g_j} = \chi_{A_j} \quad \text{such that...}$$

Sketch of the proof II: Infinite orbits admit Kakeya shadows

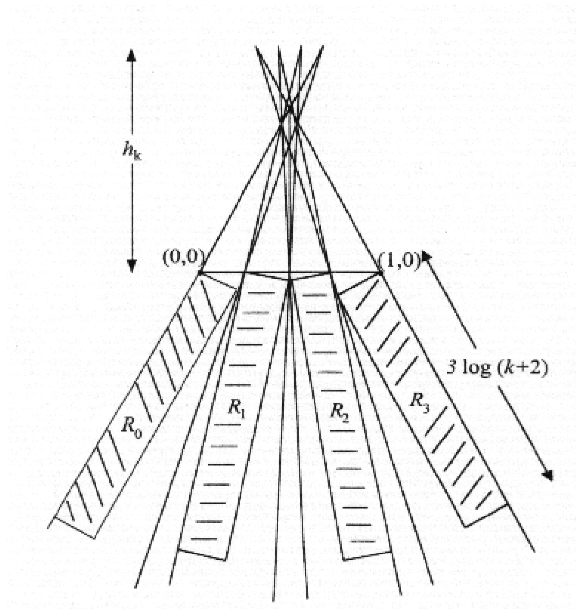
Given $\mathrm{N} \geq 1$, there exists

- A measurable set E_{N} in $\mathbb{R}^2$,
- Finite pairwise disjoint rectangles R_j in $\mathbb{R}^2$,

such that we find the inequalities $|E_{\mathrm{N}}| \lesssim \frac{1}{\mathrm{N}} \sum_j |R_j| \lesssim \frac{1}{\mathrm{N}} \sum_j \big| E_{\mathrm{N}} \cap (3R_j \setminus R_j) \big|$.



Besovitch construction



Fefferman construction

Crucial result [PR]. The orbit $\mathcal{O}_\gamma(u)$ is either finite or admits Kakeya shadows.

Given $\Lambda \subset \mathrm{G}$, set

$$L_{\Lambda,p}(\mathcal{L}(\Gamma_{\mathrm{disc}})) = \Big\{ f = \sum_{g \in \Lambda} f_g \rtimes_\gamma \lambda(g) \in L_p(\mathcal{L}(\Gamma_{\mathrm{disc}})) \Big\}.$$

Problem. Conditions on (Λ, γ, u) which yield

$$H_u \rtimes_\gamma id_{\mathrm{G}} : L_{\Lambda,p}(\mathcal{L}(\Gamma_{\mathrm{disc}})) \to L_{\Lambda,p}(\mathcal{L}(\Gamma_{\mathrm{disc}}))?$$

Theorem [P/Rogers]. $\mathcal{O}_\gamma(\Lambda, u)$ 'HD-lacunar' $\Rightarrow H_u \rtimes_\gamma id_{\mathrm{G}}$ is $L_{p,\Lambda}(\mathcal{L}(\Gamma_{\mathrm{disc}}))$-bded.

Main ingredients:

- Given $\Omega \subset \mathrm{S}^{n-1}$, set
$$M_\Omega f(x) = \sup_{w \in \Omega} \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x - t\omega)| \, dt.$$
- **Lemma**. *If* $1 < p < \infty$, $\frac{1}{q} = |1 - \frac{2}{p}|$ *and* $\frac{1}{q} < \delta < 1$
$$\Big\| \Big(\sum_{\omega \in \Omega} |H_\omega f_\omega|^2 \Big)^{\frac12} \Big\|_p \lesssim \|M_\Omega\|_{q\delta \to q\delta}^{\frac{\delta}{2}} \Big\| \Big(\sum_{\omega \in \Omega} |f_\omega|^2 \Big)^{\frac12} \Big\|_p.$$
- **Theorem** [P/Rogers]. Ω 'HD-lacunar' $\Rightarrow M_\Omega$ is $L_q(\mathbb{R}^n)$-bded for $1 < q < \infty$.

Idempotent Fourier multipliers in $\mathbb{R}$

If $\mathrm{G} = \mathbb{R} \sim \mathbb{R}_{\mathrm{disc}}$, consider

- ∘ $b_1(\xi) = (\cos 2\pi\alpha\xi - 1, \sin 2\pi\alpha\xi) \in \mathbb{R}^2 \simeq \mathbb{C}$.
- ∘ $b_2(\xi) = (\cos 2\pi\alpha\xi - 1, \sin 2\pi\alpha\xi, \cos 2\pi\beta\xi - 1, \sin 2\pi\beta\xi) \in \mathbb{R}^4 \simeq \mathbb{C}^2$.

The corresponding actions are $\gamma_\xi^1(z) = e^{2\pi i\alpha\xi} z$ and $\gamma_\xi^2(w, z) = (e^{2\pi i\alpha\xi} w, e^{2\pi i\beta\xi} z)$.

- Jodeit's th $\Rightarrow H_{\psi_1,u} : L_p(\mathbb{R}) \to L_p(\mathbb{R})$

$$\Rightarrow \Big[|\mathcal{O}_\gamma(u)| < \infty \Rightarrow H_{\psi,u} \ L_p(\mathcal{L}(\mathrm{G}))\text{-bded} \not\Rightarrow |\mathcal{O}_\gamma(u)| < \infty \Big]$$

Idempotent Fourier multipliers in $\mathbb{R}$

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- Jodeit's th $\Rightarrow \Big[|\mathcal{O}_\gamma(u)| < \infty \Rightarrow H_{\psi,u}$ $L_p(\mathcal{L}(\mathrm{G}))$-bded $\not\Rightarrow |\mathcal{O}_\gamma(u)| < \infty \Big]$

- Is the 'donut multiplier' $H_{\psi_2,u} : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ bounded for $1 < p < \infty$?

Theorem [P/Perrin/Ricard]. Let

$$\begin{aligned} \Omega &= \mathbb{T}^2 \cap \mathrm{K} && \mathrm{K} \text{ convex polyhedrum,} \\ \Sigma &= \gamma^{-1}(\mathrm{H} \cap \mathrm{K}) && \mathrm{H} \text{ non-periodic helix in } \mathbb{T}^2. \end{aligned}$$

Then T_Σ is $L_p(\mathbb{R})$-bounded $\Leftrightarrow$ T_Ω is $L_p(\mathbb{T}^2)$-bounded $\Leftrightarrow$ $\partial\Omega$ flat in $\mathbb{C}$.

Remark. 'Periodic' orbits for discrete G? Characterize $H_{\psi,u} : L_p(\mathcal{L}(\mathrm{G})) \to L_p(\mathcal{L}(\mathrm{G}))$?

Thank you!!