How non-local can a quantum state be?

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(based on previous works with M. Junge, D. Pérez-García,...)

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Quantum Mechanics: From theory to applications

- An isolated physical system is described by a density operator: $\rho : \ell_2^n \to \ell_2^n$ such that $\rho \geq 0$ and $tr(\rho) = 1$.
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IDEA: THEORY $\leftrightsquigarrow$ EXPERIMENTAL DATA

Quantum Entanglement of a bipartite state

- To describe a physical system AB formed by two physical subsystems A and B we take tensor products: $\ell_{2,A}^n \otimes_2 \ell_{2,B}^n$. Then, $\rho_{AB} \in S_{1,A}^n \otimes S_{1,B}^n$.

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- Quantum entanglement is one of the most important resources in quantum information theory. It can be quantified by the entropy of entanglement:

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- Maximally entangled state in dimension n:

$$\rho_{max} = |\psi\rangle\langle\psi|, \quad \text{where} \quad |\psi\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i \otimes e_i \in \ell_2^n \otimes \ell_2^n$$

and

$$S(\rho_{max}) = \log_2 n.$$

Quantum non-locality

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- We say that P is local ($P \in \mathcal{L}$) if it is in the convex hull of the elements of the form

$$P(a,b|x,y) = P_1(a|x)P_2(b|y) \;\; \text{for every} \;\; x,y,a,b; \;\; \text{where}$$

$P_1(a|x) \in \{0,1\}$ and $\sum_a P_1(a|x) = 1$ for every x (and similar for P_2).

How do we know if P_0 is local?

- Let's consider an arbitrary element: $M = \left(M_{x,y}^{a,b}\right)_{x,y;a,b=1}^{N,K}$ and take

$$C(M) = \sup_{P \in \mathcal{L}} |\langle M, P \rangle|, \quad \text{where} \quad \langle M, P \rangle = \sum_{x,y,a,b=1}^{N,K} M_{x,y}^{a,b} P(a,b|x,y).$$

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- Maths:

$$\sup_{Q \in \mathcal{Q}} \nu(Q) \simeq \left\| id : \ell_1^N(\ell_\infty^K) \otimes_\epsilon \ell_1^N(\ell_\infty^K) \to \ell_1^N(\ell_\infty^K) \otimes_{min} \ell_1^N(\ell_\infty^K) \right\|.$$

Quantum non-locality of a bipartite state ρ

- Given ρ we can consider the set

$$\mathcal{Q}_\rho = \left\{ \left(tr(E_x^a \otimes F_y^b \rho) \right)_{x,y;a,b=1}^{N,K}, N, K \in \mathbb{N}, \{E_x^a\}_{a=1}^K, \{F_y^b\}_{a=1}^K \right\}.$$

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- Three questions:
  1. Relation between amount of entanglement and amount of non-locality for a given state ρ.
  2. To understand the asymptotic behavior of LV_ρ for n-dimensional states.
  3. Super-activation of quantum non-locality: $LV_\rho = 1 \Rightarrow LV_{\otimes_k \rho} = 1$?

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- Junge, P. For every $\epsilon, \delta > 0$ there exists a state ρ such that

$$S(\rho) < \epsilon \ \text{ and } \ LV_\rho > \delta.$$

Quantifying non-locality

Theorem: Given a bipartite state $\rho \in S_1^n \otimes S_1^n$,

$$LV_\rho \leq \|\rho\|_{S_1^n \otimes_\pi S_1^n}.$$

In particular, $1 \leq LV_\rho \leq n$ for every n dimensional state ρ.

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Theorem (Buhrman et al.):

$$LV_{|\psi_n\rangle} \succeq \frac{n}{(\ln n)^2},$$

where $|\psi_n\rangle$ is the maximally entangled state in dimension n (note that $\pi(|\psi_n\rangle) = n$).

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Remark: Actually one can prove

$$LV_{|\varphi\rangle} \succeq \sup_{k=1,\cdots,n} \frac{\pi_k(|\varphi\rangle\langle\varphi|)}{(\ln k)^2}.$$

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Corollary: There exists a p such that

$$|\eta_p\rangle = p|\psi_n\rangle\langle\psi_n| + (1-p)\frac{1_{S_1^n \otimes S_1^n}}{n^2}$$

is local and

$$\lim_k LV_{\otimes_{i=1}^k \eta_p} = \infty.$$

What kinds of maths?

Theorem Given linear maps $S : \ell_2^n \to \ell_1(\ell_\infty)$ and $T : \ell_1(\ell_\infty) \to \ell_2^n$ such that $T \circ S = id_{\ell_2^n}$. We have that $\|T\|\|S\| \succeq \sqrt{\ln n}$.

$\Big($Furthermore this estimate is optimal even in the following non-commutative case: There exist linear maps $j : R_n \cap C_n \longrightarrow \ell_1(\ell_\infty)$ and $P : \ell_1(\ell_\infty) \to R_n \cap C_n$ such that $P \circ j = id_{\ell_2^n}$ and $\|j\|_{cb}\|P\|_{cb} \preceq \sqrt{\ln n}\Big)$.

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$$LV_{|\psi_n\rangle} \leq \left\| id : \ell_1(\ell_\infty) \otimes_\epsilon \ell_1(\ell_\infty) \to \ell_1(\ell_\infty) \otimes_{\gamma_{2_n}^*} \ell_1(\ell_\infty) \right\| \leq \frac{n}{\sqrt{\ln n}},$$

where,

$$\|x\|_{\gamma_{2_n}^*} = \sup\left\{ \left\|(u \otimes v)(x)\right\|_{\ell_2^n \otimes_\pi \ell_2^n} : \|u, v : \ell_1(\ell_\infty) \to \ell_2^n\| \leq 1 \right\}.$$

Thank you very much!