A Murray-von Neumann type classification of C^*-algebras

Chi-Keung Ng
(joint work with Ngai-Ching Wong)

The Chern Institute of Mathematics
Nankai University

Wuhan; 5th-8th June, 2012

• Types I, II, III von-Neumann algebras (Murray-von Neumann): $\mathcal{R} = \mathcal{R}_I \oplus \mathcal{R}_{II} \oplus \mathcal{R}_{III}$
v.s. Types I, II, III C^*-algebras (Cuntz-Pedersen)

• Murray-von Neumann type classification for C^*-algebras using open projections?

• $\{$open projections$\} = \{$hereditary C^*-subalgebras$\}$ and $\{$central open projections$\} = \{$ideals$\}$

• (Lin) There exists open projections p, q of a C^*-algebra A such that p and q are Murray-von Neumann equivalent but their corresponding hereditary C^*-algebras are non-isomorphic.

• Want an equivalence relation on open projections that is compatible with the Murray-von Neumann equivalence and respects the corresponding hereditary C^*-subalgebras.
No such equivalence relation was found except the one by Peligrad and Zsidó (which we only discovered in a later stage of this work).

- A is a C^*-algebra and $\mathrm{OP}(A) \subseteq \mathrm{Proj}(A^{**})$ is the set of all open projections of A.
- $\forall p \in \mathrm{Proj}(A^{**})$, we set $\mathrm{her}_A(p) := pA^{**}p \cap A$. We may sometimes write $\mathrm{her}(p)$ if A is understood.
- Let $p, q \in \mathrm{Proj}(A^{**})$. If $\exists v \in A^{**}$ s.t. $p = vv^*$, $q = v^*v$, $v^*\,\mathrm{her}_A(p)v = \mathrm{her}_A(q)$ and $v\,\mathrm{her}_A(q)v^* = \mathrm{her}_A(p)$, we say that p, q are *spatially equivalent* and denote $p \sim_{\mathrm{sp}} q$.
- If $B \subseteq A$ is a hereditary C^*-subalgebra, $p, q \in \mathrm{Proj}(B^{**})$, then $\mathrm{her}_A(p) = \mathrm{her}_B(p)$ and $p \sim_{\mathrm{sp}} q$ as elements in $\mathrm{Proj}(B^{**})$ if and only if $p \sim_{\mathrm{sp}} q$ as elements in $\mathrm{Proj}(A^{**})$.
- If $p \sim_{\mathrm{sp}} q$ and $p \in \mathrm{OP}(A)$, then $q \in \mathrm{OP}(A)$.

Proposition

(a) If $p, q \in \mathrm{OP}(A)$*, the following are equivalent.*
1. $p \sim_{\mathrm{sp}} q$.
2. $\exists$ *part. isom.* $u \in A^{**}$ *s.t.* $\mathrm{her}(q) = u^* \mathrm{her}(p) u$ *and* $\mathrm{her}(p) = u\, \mathrm{her}(q) u^*$.
3. $\exists$ *part. isom.* $w \in A^{**}$ *s.t.* $p = ww^*$ *and*

$$\{w^* r w : r \in \mathrm{OP}(A); r \leq p\} = \{s \in \mathrm{OP}(A) : s \leq q\}.$$

(b) If $\mathcal{R}$ *is a von Neumann alg. and* $p, q \in \mathrm{Proj}(\mathcal{R})$*, then* $p \sim_{\mathrm{sp}} q$ *as elements in* $\mathrm{Proj}(\mathcal{R}^{**})$ $\Leftrightarrow$ $p \sim_{\mathrm{Mv}} q$ *as elements in* $\mathrm{Proj}(\mathcal{R})$.

- Two hered. C^*-alg. $B, C \subseteq A$ are s.t.b. *spatially isomorphic* if $\exists$ part. isom. $u \in A^{**}$ such that $B = u^* C u$ and $C = u B u^*$.

Let $p, q \in \mathrm{OP}(A)$ with $p \leq q$.

• The *closure of p in q*, denoted by $\bar{p}^q$, is the smallest closed projection of her(q) that dominates p.
We say that p is *dense in q* if $\bar{p}^q = q$.
In this case, we say that her(p) is *essential* in her(q) (S. Zhang)
$\Leftrightarrow$ her(p) $\cap B \neq (0)$ if $(0) \neq B \subseteq$ her(q) hered. C^*-subalg.

• p is said to be

1. *abelian* if her(p) is a commutative C^*-algebra;
2. *C^*-finite* if for any $r, s \in \mathrm{OP}(\mathrm{her}(p))$ with $r \leq s$ and $r \sim_{\mathrm{sp}} s$, one has $\bar{r}^s = s$.

$\mathrm{OP}_{\mathcal{C}}(A)$ and $\mathrm{OP}_{\mathcal{F}}(A)$ are the set of all abelian open proj. and the set of all C^*-finite open proj. of A resp.

• p is abelian $\Rightarrow$ p is C^*-finite.

• her(p) is finite dimensional $\Rightarrow$ p is C^*-finite.

Definition

A C^*-algebra A is said to be:

i. *C^*-finite if $1 \in \mathrm{OP}_{\mathcal{F}}(A)$;*
ii. *C^*-semi-finite if any $q \in \mathrm{OP}(A) \setminus \{0\} \geq$ a proj. $p \in \mathrm{OP}_{\mathcal{F}}(A) \setminus \{0\}$;*
iii. *of Type $\mathfrak{A}$ if any $q \in \mathrm{OP}(A) \cap \mathrm{Z}(A^{**}) \setminus \{0\} \geq$ a proj. $p \in \mathrm{OP}_{\mathcal{C}}(A) \setminus \{0\}$;*
iv. *of Type $\mathfrak{B}$ if $\mathrm{OP}_{\mathcal{C}}(A) = \{0\}$ but every $q \in \mathrm{OP}(A) \cap \mathrm{Z}(A^{**}) \setminus \{0\} \geq$ a proj. $p \in \mathrm{OP}_{\mathcal{F}}(A) \setminus \{0\}$;*
v. *of Type $\mathfrak{C}$ if $\mathrm{OP}_{\mathcal{F}}(A) = \{0\}$.*

- If A is simple, then A is either of type $\mathfrak{A}$, type $\mathfrak{B}$ or type $\mathfrak{C}$.

- A C^*-algebra A is
  i. C^*-finite: $\forall$ hered. C^*-subalg. $B \subseteq A$, every hered. C^*-subalg. $C \subseteq B$ spat. isomor. to B is essent. in B;
  ii. C^*-semi-finite: every $\neq 0$ hered. C^*-subalg. of A contains $\neq 0$ C^*-finite hered. C^*-subalg;
  iii. of type $\mathfrak{A}$: every $\neq 0$ closed ideal of A contains $\neq 0$ abelian hered. C^*-subalg;
  iv. of type $\mathfrak{B}$: $\nexists \neq 0$ abelian hered. C^*-subalg. but every $\neq 0$ closed ideal contains $\neq 0$ C^*-finite hered. C^*-subalg;
  v. of type $\mathfrak{C}$: $\nexists \neq 0$ C^*-finite hered. C^*-subalg.
- A is commutative $\Rightarrow$ A is type $\mathfrak{A}$ and C^*-finite.
- $\mathcal{K}(\ell^2)$ is of type $\mathfrak{A}$, C^*-semi-finite but not C^*-finite.

Theorem

Let A and B be two strongly Morita equiv. C^*-alg.

(a) A has $\neq 0$ abelian hered. C^*-subalg. $\Leftrightarrow$ B does.

(b) A has $\neq 0$ C^*-finite hered. C^*-subalg. $\Leftrightarrow$ B does.

(c) A is of type $\mathfrak{A}$ (resp. type $\mathfrak{B}$, type $\mathfrak{C}$ or C^*-semi-finite) $\Leftrightarrow$ B is of type $\mathfrak{A}$ (resp. type $\mathfrak{B}$, type $\mathfrak{C}$ or C^*-semi-finite).

- A is of type $\mathfrak{A}$ $\Leftrightarrow$ A is *discrete*, in the sense of Peligrad-Zsidó: any $\neq 0$ hered. C^*-subalg. contains $\neq 0$ abelian hered. C^*-subalg.
- A is C^*-semi-finite $\Leftrightarrow$ any $\neq 0$ closed ideal contains $\neq 0$ C^*-finite hered. C^*-subalg.

Corollary

Let A be a C^*-algebra with real rank zero.

(a) A is of type $\mathfrak{A}$ $\Leftrightarrow$ any $q \in \mathrm{Proj}(A) \setminus \{0\} \geq$ an abelian $p \in \mathrm{Proj}(A) \setminus \{0\}$.

(b) A is of type $\mathfrak{B}$ $\Leftrightarrow$ $\nexists \neq 0$ abelian proj. but every $q \in \mathrm{Proj}(A) \setminus \{0\} \geq$ a C^*-finite $p \in \mathrm{Proj}(A) \setminus \{0\}$.

(c) A is of type $\mathfrak{C}$ $\Leftrightarrow$ $\nexists \neq 0$ C^*-finite projection.

(d) A is C^*-semi-finite $\Leftrightarrow$ every $q \in \mathrm{Proj}(A) \setminus \{0\} \geq$ a C^*-finite $p \in \mathrm{Proj}(A) \setminus \{0\}$.

• Let p be a projection in a von Neumann algebra $\mathcal{R}$. Then p is finite if and only if it is C^*-finite.

Proposition

Let $\mathcal{R}$ be a von Neumann algebra.

(a) $\mathcal{R}$ is of type $\mathfrak{A}$ $\Leftrightarrow$ $\mathcal{R}$ is a type I von Neumann alg.

(b) $\mathcal{R}$ is of type $\mathfrak{B}$ $\Leftrightarrow$ $\mathcal{R}$ is a type II von Neumann alg.

(c) $\mathcal{R}$ is of type $\mathfrak{C}$ $\Leftrightarrow$ $\mathcal{R}$ is a type III von Neumann alg.

(d) $\mathcal{R}$ is C^*-semi-finite $\Leftrightarrow$ $\mathcal{R}$ is a semi-finite von Neumann alg.

Corollary

Suppose that A is of type $\mathfrak{A}$ (resp. type $\mathfrak{B}$, type $\mathfrak{C}$ or C^*-semi-finite).

(a) Any hered. C^*-subalg. of A has the same property.

(b) If $A \subseteq B$ hered. C^*-subalg. generating an essential ideal $I \subseteq B$, then B has the same property.

- All the types are respected under taking multiplier algebras.

Proposition

(a) Any type I C^*-algebra is of type $\mathfrak{A}$.

(b) A is of type I if and only if every primitive quotient of A is of type $\mathfrak{A}$.

- $\mathcal{B}(\ell^2)$ is of type $\mathfrak{A}$ but is not a type I C^*-algebra.

- If A is a simple C^*-algebra of type $\mathfrak{A}$, then $A = \mathcal{K}(H)$ for some Hilbert space H. If, in addition, A is C^*-finite, then $A = M_n$ for some positive integer n.

Proposition

(a) If A is finite, in the sense of Cuntz-Pedersen, then A is C^*-finite.

(b) If A is semi-finite (resp. of type II), in the sense of Cuntz-Pedersen, then A is C^*-semi-finite (resp. of type $\mathfrak{B}$).

• If A is an inf. dim. simple C^*-algebra with a faithful tracial state, then A is of type $\mathfrak{B}$. In particular, if Γ is an infin. discrete group s.t. $C^*_r(\Gamma)$ is simple, then $C^*_r(\Gamma)$ is of type $\mathfrak{B}$.

• Every simple AF algebra which is not of the form $\mathcal{K}(H)$ is of type $\mathfrak{B}$.

Proposition

(a) If A is of type $\mathfrak{C}$, then it is of type III, in the sense of Cuntz-Pedersen.

(b) If A has real rank zero and is purely infinite, in the sense of Kirchberg-Rørdam, A is of type $\mathfrak{C}$.

(c) If A is a separable purely infinite C^*-algebra with stable rank one, then A is of type $\mathfrak{C}$.

• For any *AF*-algebra *B*, the C^*-algebra $\mathcal{O}_2 \otimes B$ is of type $\mathfrak{C}$. Note that one may replace $\mathcal{O}_2$ with any unital, simple, separable, purely infinite, nuclear C^*-algebra.

Theorem

Let A be a C^*-algebra.

(a) $\exists$ a largest type $\mathfrak{A}$ (resp. type $\mathfrak{B}$, type $\mathfrak{C}$ and C^*-semi-finite) hered. C^*-subalg $J_{\mathfrak{A}}$ (resp. $J_{\mathfrak{B}}$, $J_{\mathfrak{C}}$ and $J_{\mathfrak{sf}}$) of A, which is also an ideal of A.

(b) $J_{\mathfrak{A}}$, $J_{\mathfrak{B}}$ and $J_{\mathfrak{C}}$ are mutually disjoint, and $J_{\mathfrak{A}} + J_{\mathfrak{B}} + J_{\mathfrak{C}}$ is an essent. ideal of A.
If $e_{\mathfrak{A}}, e_{\mathfrak{B}}, e_{\mathfrak{C}} \in \mathrm{OP}(A) \cap Z(A^{**})$ with $J_{\mathfrak{A}} = \mathrm{her}(e_{\mathfrak{A}})$, $J_{\mathfrak{B}} = \mathrm{her}(e_{\mathfrak{B}})$ and $J_{\mathfrak{C}} = \mathrm{her}(e_{\mathfrak{C}})$, then

$$1 = \overline{e_{\mathfrak{A}} + e_{\mathfrak{B}}}^{1} + e_{\mathfrak{C}}.$$

(c) $J_{\mathfrak{A}} + J_{\mathfrak{B}}$ is an essent. ideal of $J_{\mathfrak{sf}}$. If $e_{\mathfrak{sf}} \in \mathrm{OP}(A)$ with $J_{\mathfrak{sf}} = \mathrm{her}(e_{\mathfrak{sf}})$, then $e_{\mathfrak{sf}} = \overline{e_{\mathfrak{A}}}^{e_{\mathfrak{sf}}} + e_{\mathfrak{B}}$.

- Set $J^\perp$ to be the ideal $\{a \in A : aJ = (0)\}$

a. If $J_{postlim}$ is the largest type I closed ideal of A, then $J_{\mathfrak{A}}^\perp = J_{postlim}^\perp$ is the largest anti-liminary hered. C^*-subalg. of A, and it contains $J_{\mathfrak{B}} + J_{\mathfrak{C}}$ as an essent. ideal.

b. $J_{\mathfrak{sf}}^\perp = (J_{\mathfrak{A}} + J_{\mathfrak{B}})^\perp = J_{\mathfrak{C}}$.

c. $J_{\mathfrak{A}}^\perp \cap J_{\mathfrak{sf}} = J_{\mathfrak{B}}$.

Theorem

Let A be a C^*-algebra.

(a) $A/J_{\mathfrak{C}}$ is C^*-semi-finite.

(b) If A is C^*-semi-finite, then $A/J_{\mathfrak{B}}$ is of type $\mathfrak{A}$.

- If A and B are str. Morita equiv, the closed ideal of B that corresp. to $J_{\mathfrak{A}}^A$ (resp. $J_{\mathfrak{B}}^A$, $J_{\mathfrak{C}}^A$ and $J_{\mathfrak{sf}}^A$) under the str. Morita equiv. is precisely $J_{\mathfrak{A}}^B$ (resp. $J_{\mathfrak{B}}^B$, $J_{\mathfrak{C}}^B$ and $J_{\mathfrak{sf}}^B$).

• We say that a property $\mathcal{P}$ concerning C^*-algebras is *hereditarily stable* if $\forall$ C^*-alg A having prop. $\mathcal{P}$, all hered. C^*-subalg. of A will have prop. $\mathcal{P}$.

• A sequence $\{\mathcal{P}_1, ..., \mathcal{P}_n\}$ of hered. stable properties is s.t.b. *compatible* if $\mathcal{P}_{i-1}$ is stronger than $\mathcal{P}_i$ for $i = 1, ..., n$, where $\mathcal{P}_0$ means "the C^*-algebra is zero".

• Let $\{\mathcal{P}_i\}_{i=1,...n}$ be a seq. of compat. hered. stable prop. and set $\mathcal{P}_{n+1}$ to be: "the C^*-algebra contains a zero element" (i.e. a tautology). We say that A is of *type* $\mathcal{T}_i^{\mathcal{P}}$ $(i = 1, ..., n+1)$ if

> $\nexists \neq 0$ *hered.* C^**-subalg. of* A *having prop.* $\mathcal{P}_{i-1}$*, but any* $\neq 0$ *closed ideal of* A *contains* $\neq 0$ *hered.* C^**-alg. with property* $\mathcal{P}_i$*.*

• Define $P_i(A) := \{e \in \mathrm{proj}(A) : \mathrm{her}(e) \text{ has prop. } \mathcal{P}_i\}$.

Theorem

Let $\{\mathcal{P}_i\}_{i=1,\dots,n}$ be a seq. of compat. hered. stable prop.

(a) If A is simple, then A is of type $\mathcal{T}_j^{\mathcal{P}}$ for exactly one $j = 1, \dots, n+1$.

(b) If A is str. Morita equiv. to a C^*-alg. of type $\mathcal{T}_i^{\mathcal{P}}$, then A is of type $\mathcal{T}_i^{\mathcal{P}}$.

(c) If A is a hered. C^*-subalg. of a C^*-alg. of type $\mathcal{T}_i^{\mathcal{P}}$, then A is of type $\mathcal{T}_i^{\mathcal{P}}$.

(d) If A contains a hered. C^*-subalg. that generates an essen. ideal of type $\mathcal{T}_i^{\mathcal{P}}$, then A is of type $\mathcal{T}_i^{\mathcal{P}}$.

(e) If A has RR0, then A is of $\mathcal{T}_i^{\mathcal{P}}$ if and only if $P_{i-1}(A) = \{0\}$ and any $q \in \mathrm{Proj}(A) \setminus \{0\} \geq$ a proj. $p \in P_i(A) \setminus \{0\}$.

Theorem

(cont.)

(f) $\exists$ *a largest type* $\mathcal{T}_i^{\mathcal{P}}$ *hered.* C^**-subalg.* $J_i \subseteq A$*, which is an ideal of A s.t.* $J_1, \ldots, J_{n+1}$ *are mutually disjoint.*

(g) If $e_i \in \mathrm{OP}(A)$ *with* $J_i = \mathrm{her}(e_i)$*, then* $\overline{\sum_{i=1}^n e_i}^1 + e_{n+1} = 1$*, and* $J_1 + \ldots + J_{n+1}$ *is essen. ideal of A.*

(h) Str. Morita equiv. respects J_i*.*

(i) If every $\neq 0$ *closed ideal of A contains* $\neq 0$ *hered.* C^**-subalg. having* $\mathcal{P}_i$*, then every* $\neq 0$ *closed ideal of* A/J_i *contains* $\neq 0$ *hered.* C^**-subalg. having* $\mathcal{P}_{i-1}$*.*

Corollary

(a) *If A is str. Morita equiv. to a discrete (resp. type II, type III or semi-finite) C^*-alg, then A has the same property.*

(b) *If A is a hered. C^*-subalg. of a discrete (resp. type II, type III or semi-finite) C^*-alg, then A has the same property.*

(c) *If A contains an essen. hered. C^*-subalg that is discrete (resp. of type II, of type III or semi-finite), then A also has the same property.*

(d) *The sum of the largest discrete closed ideal, the largest type II closed ideal and the largest type III closed ideal of A (all of them exist) is essential in A.*

Proposition

Let $\{\mathcal{P}_i\}_{i=1,...,n}$ and $\{\mathcal{P}'_i\}_{i=1,...,n}$ be two seq. of compat. hered. stable prop. Then "type $\mathcal{T}_i^{\mathcal{P}}$ = type $\mathcal{T}_i^{\mathcal{P}'}$" $(i = 1, ..., n+1)$ $\Leftrightarrow$ any $\neq 0$ C^*-alg with $\mathcal{P}_i$ contains $\neq 0$ hered. C^*-subalg. with $\mathcal{P}'_i$ and vice versa $(i = 1, ..., n)$.

Reference:

C.K. Ng and N.C. Wong, A Murray-von Neumann type classification of C^*-algebras, preprint (arXiv:1112.1455).

Thanks