An H_1-BMO duality for Markov Semigroups of Operators

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Question

$(\mathcal{M}, \tau)$: finite von Neumann algebra;
$L_p(\mathcal{M})$: noncommutative L_p spaces, i.e.

$$L_p(\mathcal{M}) = \{x \in L_0(\mathcal{M}), \|x\|_p = (\tau|x|^p)^{\frac{1}{p}} < \infty\}.$$

What are noncommutative analogues of classical (real) BMO and H_1 spaces so that

$$(H_1)^* = BMO, \quad [BMO, H_1]_{\frac{1}{p}} = L_p(\mathcal{M}), 1 < p < \infty?$$

Work on Noncommutative Martingales: Pisier/Xu, Junge/Xu, Musat, Randrianantoanina/Parcet, Bekjan-Chen-Perrin-Yin, Hong, etc. ;

Classical BMO and Hardy spaces

$(\mathcal{M},\tau)=(L_\infty(\mathbb{R}),dx)$.
$f\in L_1^{loc}(\mathbb{R})$ $f_I=\frac{1}{|I|}\int_I fdx$.

$$\|f\|^2_{\mathrm{BMO}(\mathbb{R})}=\sup_{I\subset\mathbb{R}}\frac{1}{|I|}\int_I|f(x)-f_I|^2dx,=\sup_{I\subset\mathbb{R}}\{(|f|^2)_I-|f_I|^2\}.$$

$\mathrm{BMO}(\mathbb{R})=\{f\in L_1(\mathbb{R},\frac{dx}{1+x^2});\|f\|_{BMO(\mathbb{R})}<\infty\}$.

$$\begin{aligned}
G(f)(x) &= \left(\int_0^\infty|\nabla P_t(f)|^2tdt\right)^{\frac12},\\
S(f)(x) &= \left(\int_{\{(y,t):|y-x|<t\}}|\nabla P_t(f)|^2tdy\frac{dt}{t^n}\right)^{\frac12}.\\
\|f\|_{H_1} &= \|G(f)\|_{L_1}+\|f\|_{L_1}\simeq\|S(f)\|_{L_1}+\|f\|_{L_1}.\\
H_1(\mathbb{R}) &= \{f\in L_1(\mathbb{R});\|f\|_{H_1}<\infty\}.
\end{aligned}$$

Garnett, Koosis, Varopoulos, Duong/Yan... Replacing $\frac{1}{|I|}\int_I\cdot$ by T_t,
$\|f\|^2_{\mathrm{BMO}(\mathbb{R})}\simeq\sup_t\|T_t|f-T_tf(\cdot)|^2\|_\infty=\sup_t\||T_t|f|^2-|T_tf|^2\|_\infty$.

BMO and Hardy spaces on metric spaces

(M, μ)= nice metric space (doubling measure+Davies-Gaffney property).

$(T_t)_t$= semigroups of operators with fast decreasing kernels.

$$\|f\|^2_{\mathrm{BMO(M)}} = \sup_{\mathrm{ball} I \subset M} \frac{1}{|I|} \int_I |f - T_{r_I^m} f|^2 dx.$$

$$S(f)(x) = \Big(\int_{\{(y,t):|y-x|<t^{\frac{1}{m}}\}} |\partial_t P_t(f)|^2 t dy \frac{dt}{t^n} \Big)^{\frac{1}{2}}.$$

$$\|f\|_{H_1(M)} = \|S(f)\|_{L_1} + \|f\|_{L_1}.$$

$H_1(M)$= $\{f \in L_1(M); \|f\|_{H_1} < \infty\}$.

(Duong/Yan, J. AMS 2005)

$$(H_1(M))^* = BMO(M).$$

Markov Semigroups of Operators

$\mathcal{M}$: finite von Neumann algebra,
$(T_t)_{t\geq 0}$: a semigroup of operators on $\mathcal{M}$,
We say $(T_t)_t$ is Markov, if

- T_t are normal completely contractions on M.
- T_t are symmetric i.e. $\tau((T_t f)g^*) = \tau(f(T_t g)^*)$ for $f, g \in L^1(\mathcal{M}) \cap \mathcal{M}$.
- $T_t(1) = 1$
- $T_t(f) \to f$ in the w^* topology for $f \in \mathcal{M}$.

Note the first and the third assumption implies T_t is completely positive, i.e. $T_t(f) \geq 0$, for any $f \geq 0 \in \mathcal{M} \otimes \mathcal{K}(\ell_2)$.

Infinitesimal generator: $L = -\frac{\partial T_t}{\partial t}|_{t=0}$; $T_t = e^{-tL}$.

Litterwood-Paley theory by E. Stein,...., Le Merdy-Junge-Xu.

BMO associated with $(T_t)_t$

$(\mathcal{M}, \tau)$: finite von Neumann algebra;
$(T_t)_t$: Markov semigroup of operators
For $f \in L^2(\mathcal{M})$,

$$\begin{aligned} \|f\|_{BMO(T)} &= \sup_t \|T_t|f - T_t f|^2\|^{\frac{1}{2}}. \\ \|f\|_{bmo(T)} &= \sup_t \|T_t|f|^2 - |T_t f|^2\|^{\frac{1}{2}}. \\ BMO(T) &= \{f \in L^2(\mathcal{M}); \|f\|_{BMO(T)}, \|f^*\|_{BMO(T)} < \infty\}. \end{aligned}$$

(Junge-M.)

$$[BMO(T), L_1^0(\mathcal{M})]_{\frac{1}{p}} = L_p^0(\mathcal{M}).$$

$$\|L^{is}\|_{\mathcal{M} \to BMO(T)} < c.$$

(Junge-M-Parcet)
Boundedness of fourier multipliers from $\mathcal{M}$ to bmo(T) $(BMO(T))$.

$\Gamma = \partial^* \partial$ associated with L

$\mathcal{M} = L^\infty(\mathbb{R})$,
$L = \Delta = \partial^2 x$;

$$2\partial f^* \partial g = \triangle(f^* g) - \triangle f^* g - f^* \triangle g.$$

For $T_t = e^{-tL}$, set

$$2\Gamma(f, g) = -L(f^* g) + L(f^*) g + f^* L(g);$$
$$2\Gamma_2(f, g) = -L(\Gamma(f, g)) + \Gamma(L(f), g) + \Gamma(f, L(g)).$$

analogues of $\partial f^* \partial g, \partial^2 f^* \partial^2 g$

$\Gamma(f, f) \geq 0$ iff $|T_t f|^2 \leq T_t |f|^2$.

We say T_t satisfies $\Gamma_2 \geq 0$ if $\Gamma_2(f, f) \geq 0$.

P. A. Meyer, D. Bakry, M.Emery, X. D. Li, F. Baudoin-N. Garofalo, etc.

Noncommutative Riesz transforms; Quantumn metric spaces;..

H^1 space associated with $(T_t)_t$

Recall we replace $\frac{1}{|I|}\int_I \cdot$ by T_t, $\int_{\{(y,t),|y-x|<t\}} \cdot dy \frac{dt}{t^n}$ by $\int_0^\infty T_t \cdot dt$ and replace $|\nabla \cdot|^2$ by $\Gamma(\cdot,\cdot)$.
f$\in L_1(\mathcal{M})$.

$$S(f) = (\int_0^\infty T_t\Gamma(T_tf, T_tf)dt)^{\frac{1}{2}}$$

$$G(f) = (\int_0^\infty \Gamma(T_tf, T_tf)dt)^{\frac{1}{2}}.$$

Let

$$\|f\|_{H_1^S(T)} = \|S(f)\|_{L_1(\mathcal{M})} + \|f\|_{L_1(\mathcal{M})}.$$

Example $\mathcal{M} = L^\infty(\mathbb{R})$, $T_t(f) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{|x|^2}{4t}} * f$.
$H_1^S(T) = H_1(\mathbb{R})$.
Only show the good side, difficulties: geometric tools

Main results

Theorem 1 Let (T_t) be a Markov semigroup of operators satisfying $\Gamma_2 \geq 0$. Then

$$BMO(T) \subset (H_1^S(T))^*$$

i.e.

$$|\tau(f^* g)| \leq c \|f\|_{H_1^S(T)} \|g\|_{BMO(T)},$$

for all $f, g \in L_2(\mathcal{M})$.

Examples:

$(T_t)_t = e^{t\triangle}$: Heat semigroups generated by the Laplace-Beltrami operator on a complete Riemannian manifold with nonnegative Ricci curvature;

$(T_t)_t = e^{t(\triangle - \nabla\phi \cdot \nabla)}$:Ornstein-Uhlenbeck semigroups on complete Riemannian manifold with Ricci$+\nabla^2\phi \geq 0$;

$(T_t)_t =$Markov semigroups of operators on Group von Neumann algebras;

Examples on discrete groups

Consider ϕ a real valued function on a discrete group G. Define

$$\begin{aligned} L(\lambda_g) &= \phi(g)\lambda_g, \\ T_t(\lambda_g) &= e^{-t\phi(g)}\lambda_g. \end{aligned}$$

Then $(T_t)_t$ is a Markov semigroup of operators on $\mathcal{M} = L_\infty(\mathbb{G})$ if
$\phi(1) = 0, \phi(g) = \phi(g^{-1})$
and ϕ is conditionally negative,

$$\sum_{g,h} \overline{a_g} a_h \phi(g^{-1}h) \leq 0$$

for any complex numbers a_g with $\sum_g a_g = 0$.
It is easy to compute by the definition that

$$\begin{aligned} \Gamma(\sum_g a_g\lambda_g) &= \sum_{g,h} \bar{a}_g a_h K_\phi(g,h)\lambda_{g^{-1}h}, \\ \Gamma_2(\sum_g a_g\lambda_g) &= \sum_{g,h} \bar{a}_g a_h K_\phi^2(g,h)\lambda_{g^{-1}h}, \end{aligned}$$

with $K_\phi(g,h) = \frac{-\phi(gh^{-1})+\phi(g)+\phi(h)}{2}$. $\Gamma_2 \geq 0$ is satisfied.

Main results

Theorem 2 Assume, in addition, that there exist constants c_1, c_2 such that
(i) $\|tLT_t f\|_1 \leq c_1 \|f\|_1$, for all $t > 0$ and $f \in L_1(\mathcal{M})$.
(ii) Let $M_t = \frac{1}{t}\int_0^t T_s ds$

$$\|(M_{8t}|T_t f|^2)^{\frac{1}{2}}\|_{L_1(\mathcal{M})} \leq c_2 \|f\|_{L_1(\mathcal{M})},$$

for all $t > 0$ and $f \in L_1^+(\mathcal{M})$. Then

- $(H_1^S(T))^* = BMO(T)$.
- $[BMO(T), H_1^S(T)]_{\frac{1}{p}} = L_p^0(\mathcal{M})$.
- $bmo(T) = BMO(T)$.
- $\|G(f)\|_{L_1(\mathcal{M})} \simeq \|S(f)\|_{L_1(\mathcal{M})}$.

Examples: $(T_t)_t = e^{t\triangle}$: Heat semigroups generated by the Laplace-Beltrami operator on a complete Riemannian manifold with nonnegative Ricci curvature;

Examples on discrete groups

G : discrete group
ϕ: $G \to \mathbb{R}$, $\phi(e) = 0, \phi(g) = \phi(g^{-1})$ and conditionally negative .

$T_t(\lambda_g) = e^{\phi(g)t}\lambda_g$;
$\mathbb{R}[G]$: the algebra of all real valued bounded functions on G.

$$\langle \sum_g a_g \delta_g, \sum_h b_h \delta_h \rangle_\phi = \sum_{g,h} a_g a_h K_\phi(g,h)$$

with $K_\phi(g,h) = \frac{-\phi(gh^{-1}) + \phi(g) + \phi(h)}{2}$ defines a semi-inner product on $\mathbb{R}[G]$.
$N_\phi = \{x \in \mathbb{R}[G], \langle x, x \rangle_\phi = 0\}$,

$(T_t)_t$ satisfies all assumptions of Theorem 2 if

$$\dim \mathbb{R}[G]/N_\phi < \infty,$$

by Junge's reduction trick.