On application of Orlicz spaces to Statistical Physics.

Wladyslaw Adam Majewski

Instytut Fizyki Teoretycznej i Astrofizyki, UG
ul. Wita Stwosza 57, 80-952 Gdańsk, Poland;
joint work with Louis E. Labuschagne

- MOTIVATION and OUTLINE:

- *To indicate reasons why (classical as well as non-commutative) Orlicz spaces are emerging in the theory of (classical and quantum) Physics*

- When a physicist knows that a certain quantity is an observable?

- In any answer: an observable is known when also a function of this observable is known.

- This feature of observables was, probably, a motivation for Newton to develop calculus and to use it in his laws of motion.

- Within the probability calculus, the same question will imply: we wish to know the average of $< u >_m$ as well as $< F(u) >_m$, at least, for a large class of functions F.

- Assume that F has the Taylor expansion $F(x) = \sum_i c_i x^i$. Our demands mean that $< F(u) >_m = \sum_i c_i < u^i >_m$ should be well defined. However, this implies that "regular" observables should have all moments finite.

- Define states of a system

$$\mathcal{S}_m = \{f \in L^1(m) : f > 0 \quad \mu - a.s., E(f) = 1\}.$$

- $E(f) \equiv < f >_m$ stands for $\int f(x) dm(x)$.

- Fix $f \in \mathcal{S}_m$ and take a real random variable u on $(X, \Sigma, f dm)$. Define moment generating functions:

$$\hat{u}_f(t) = \int exp(tu) f dm, \qquad t \in \mathbb{R}$$

- Denote by L_f the set of all random variables such that $\hat{u}_f$ is well defined in a neighborhood of the origin 0, and the expectation of u is zero.

- L_f is actually the **Orlicz space** based on an exponentially growing function $cosh - 1$

- Entropy

  1. $H(f) = -\int f(x) ln f(x) d\mu$, $f \in \mathcal{S}_\mu$, for the classical (continuous) case;
  2. $S(\varrho) = -Tr \varrho ln \varrho$, ϱ a density matrix, for the quantum case.

- The problem is that both definitions can lead to divergences.

- The set of "good" density matrices $\{\varrho : S(\varrho) < \infty\}$ is a meager set only (we assume that the dimension of the underlying Hilbert space is infinite!).

- Thus we run into serious problems with the explanation of the phenomenon of return to equilibrium and with the second law of thermodynamics (entropy should be a state function which is increasing in time).

- To solve both outlined above problems we propose to replace the pair of Banach spaces

$$< L^{\infty}(X, \Sigma, m), L^{1}(X, \Sigma, m) > \tag{1}$$

- by the pair of Orlicz spaces (or equivalent pairs)(at least for finite measure space).

$$< L^{cosh-1}, L^{(\cdot)ln(\cdot+\sqrt{1+(\cdot)^2})-\sqrt{1+(\cdot)^2}+1} > . \tag{2}$$

- Consequently, we propose **the new rigorous approach** for description of statistics of regular statistical systems **having this advantage that statistics as well as thermodynamics are well settled down**

Orlicz spaces

- **Definition 1.** *Let* $\psi : [0, \infty) \to [0, \infty]$ *be an increasing and left-continuous function such that* $\psi(0) = 0$. *Suppose that on* $(0, \infty)$ ψ *is neither identically zero nor identically infinite. Then the function* Ψ *defined by*

$$\Psi(s) = \int_0^s \psi(u)du, \qquad (s \geq 0) \tag{3}$$

is said to be a Young's function.

- $x \mapsto cosh(x) - 1$, $x \mapsto xln(x + \sqrt{1 + x^2}) - \sqrt{1 + x^2} + 1$ are Young's functions while $x \mapsto xlnx$ not. As we will be interested in nice Young's functions, in the sequel, we will always assume that these functions are continuous, positive and equal to 0 only for $x = 0$.

- **Definition 2.** *1. A Young's function Ψ is said to satisfy the Δ_2-condition if there exist $s_0 > 0$ and $c > 0$ such that*

$$\Psi(2s) \leq c\Psi(s) < \infty, \qquad (s_0 \leq s < \infty). \tag{4}$$

2. A Young's function Φ is said to satisfy ∇_2-condition if there exist $x_0 >$ and $l > 1$ such that

$$\Phi(x) \leq \frac{1}{2l}\Phi(lx) \tag{5}$$

for $x \geq x_0$.

- It is easy to verify that the Young's function, given prior to Definition 2, $x \mapsto xln(x + \sqrt{1 + x^2}) - \sqrt{1 + x^2} + 1$, $(x \mapsto cosh(x) - 1)$, satisfies the Δ_2-condition (∇_2-condition, respectively).

- **Definition 3.** *Let Ψ be a Young's function, represented as in (3) as the integral of ψ. Let*

$$\phi(v) = \inf\{w : \psi(w) \geq v\}, \qquad (0 \leq v \leq \infty). \tag{6}$$

Then the function

$$\Phi(t) = \int_0^t \phi(v)dv, \qquad (0 \leq t \leq \infty) \tag{7}$$

is called the complementary Young's function of Ψ.

- We note that if the function $\psi(w)$ is continuous and increasing monotonically then $\phi(v)$ is a function exactly inverse to $\psi(w)$.

- Define (another Young's function)

$$xln(x + \sqrt{1 + x^2}) - \sqrt{1 + x^2} + 1 = \int_0^x arsinh(v)dv. \tag{8}$$

- **Corollary 4.** $xln(x + \sqrt{1 + x^2}) - \sqrt{1 + x^2} + 1$ *and* $coshx - 1$ *are complementary Young's functions.*

- Let L^0 be the space of measurable functions on some σ-finite measure space (X, Σ, m). We will always assume, that the considered measures have the finite subset property, i.e. $E \in \Sigma$, $m(E) > 0$ implies the existence of $F \in \Sigma$ such that $F \subset E$ and $0 < m(F) < \infty$.

- **Definition 5.** *The Orlicz space L^Ψ associated with Ψ is defined to be the set*

$$L^\Psi \equiv L^\Psi(X, \Sigma, m) = \{f \in L^0 : \Psi(\lambda|f|) \in L^1 \quad \textit{for some} \quad \lambda = \lambda(f) > 0\}. \tag{9}$$

- Luxemburg-Nakano norm

$$\|f\|_\Psi = \inf\{\lambda > 0 : \|\Psi(|f|/\lambda)\|_1 \leq 1\}.$$

- An equivalent - Orlicz norm, for a pair (Ψ, Φ) of complementary Young's functions is given by

$$\|f\|_\Phi = \sup\{\int |fg|dm : \int \Psi(|g|)dm \leq 1\}.$$

- L_p-spaces are nice examples of Orlicz spaces. Zygmund spaces:

- – $LlogL$ is defined by the following Young's function

$$slog^+s = \int_0^s \phi(u)du$$

  where $\phi(u) = 0$ for $0 \leq u \leq 1$ and $\phi(u) = 1 + logu$ for $1 < \infty$, where $log^+x = max(logx, 0)$

  – L_{exp} is defined by the Young's function

$$\Psi(s) = \int_0^s \psi(u)du,$$

  where $\psi(0) = 0$, $\psi(u) = 1$ for $0 < u < 1$, and $\psi(u)$ is equal to e^{u-1} for $1 < u < \infty$. Thus $\Psi(s) = s$ for $0 \leq s \leq 1$ and $\Psi(s) = e^{s-1}$ for $1 < s < \infty$.

- To understand the role of Zygmund spaces the following result will be helpful

Theorem 6. *Take $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ for the measure space with $\mu(\mathbb{R}) = 1$. The continuous embeddings*

$$L^\infty \hookrightarrow L_{exp} \hookrightarrow L^p \hookrightarrow LlogL \hookrightarrow L^1 \tag{10}$$

hold for all p satisfying $1 < p < \infty$. Moreover, L_{exp} may be identified with the Banach space dual of $LlogL$.

- More generally, for a pair (Ψ, Φ) of complementary Young's functions with the function Ψ satisfying Δ_2-condition there is the following relation $(L^\Psi)^* = L^\Phi$.

- Finally, we will write $F_1 \succ F_2$ if and only if $F_1(bx) \geq F_2(x)$ for $x \geq 0$ and some $b > 0$, and we say that the functions F_1 and F_2 are equivalent, $F_1 \approx F_2$, if $F_1 \prec F_2$ and $F_1 \succ F_2$.

- **Example 7.** *Consider, for* $x > 0$

  - $F_1(x) = xln(x + \sqrt{1 + x^2}) - \sqrt{1 + x^2} + 1 = \int_0^x ln(s + \sqrt{1 + x^2})ds$,
  - $F_2 = kxlnx = k \int_0^x (lns + 1)ds$, $k > e$.

  Then $F_1 \succ F_2$.

- **Remark 8.** *1. Recall,* $x \mapsto xlnx$ *is not a Young's function. Therefore, it is difficult to speak about Orlicz space*L^{xlnx}.
  2. *If* $\Psi \succ F$, Ψ *is a Young's function satisfying* Δ_2*-condition, the function* F *is bounded below by* $-c$*, then for* $f \in L^{\Psi}$ *the integral* $\int F(f)(u)dm(u)$ *is finite provided that the measure* m *is finite.*

- **Proposition 9.** *The function* $\mathbb{R}^+ \ni t \mapsto cosht - 1$ *and* $\mathbb{R}^+ \mapsto e^t - t - 1 \equiv \Phi(t)$ *are equivalent, i.e.*

$$cosht - 1 \approx e^t - t - 1. \tag{11}$$

- **Theorem 10.** *Let* Φ_i*,* $i = 1, 2$ *be a pair of equivalent Young's function. Then* $L^{\Phi_1} = L^{\Phi_2}$*.*

- These results lead to

Corollary 11. *For finite measure spaces* $(\mathcal{X}, \Sigma, \mu)$ *one has*

$$L^{cosh-1} = L_{exp}. \tag{12}$$

Regular classical systems

- Let $\{\Omega, \Sigma, \nu\}$ be a measure space; ν will be called the reference measure. The set of densities of all the probability measures equivalent to ν will be called the state space $\mathcal{S}_\nu$, i.e.

$$\mathcal{S}_\nu = \{f \in L^1(\nu) : f > 0 \quad \nu - a.s., E(f) = 1\}, \tag{13}$$

$E(f) \equiv \int f d\nu$. $f \in \mathcal{S}_\nu$ implies that $f d\nu$ is a probability measure.

- **Definition 12.** *The classical statistical model consists of the measure space* $\{\Omega, \Sigma, \nu\}$*, state space* $\mathcal{S}_\nu$*, and the set of measurable functions* $L^0(\Omega, \Sigma, \nu)$.

- We defined on $(\Omega, \Sigma, f d\nu)$

$$\hat{u}_f(t) = \int exp(tu) f d\nu, \qquad t \in \mathbb{R}. \tag{14}$$

- and

Definition 13. *The set of all random variables on (Ω, Σ, ν) such that for a fixed $f \in S_\nu$*

1. *$\hat{u}_f$ is well defined in a neighborhood of the origin 0,*
2. *the expectation of u is zero,*

will be denoted by $L_f \equiv L_f(f \cdot \nu)$ and called the set of regular random variables (these conditions imply that all moments are finite!).

- It was proved

Theorem 14. *(Pistone-Sempi) L_f is the closed subspace of the Orlicz space $L^{\cosh -1}(f \cdot \nu)$ of zero expectation random variables.*

- Note that there is the relation $\succ$ between the Young's function $xln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + 1$ and the entropic function $c \cdot xlnx$ where c is a positive number. Consequently, the condition $f \in L^{xln}(f \cdot \nu)$ guarantees (for finite measure case) that the continuous entropy is well defined.

- **Corollary 15.**

$$< L^{cosh-1}, L^{(\cdot)ln(\cdot+\sqrt{1+(\cdot)^2})-\sqrt{1+(\cdot)^2}+1} >$$

or equivalently

$< L_{exp}, LlogL >$

provides the proper framework for the description of classical regular statistical systems (based on probability measures).

Non-commutative Orlicz spaces

- Let Φ be a given Young's function. $\mathcal{M}$ a semifinite von Neumann algebra equipped with an fns (faithful normal semifinite) trace τ,
- The space of all τ-measurable operators $\widetilde{\mathcal{M}}$ (equipped with the topology of convergence in measure) plays the role of L^0 .
- Kunze defined the associated noncommutative Orlicz space to be

$$L_\Phi^{ncO} = \cup_{n=1}^{\infty} n\{f \in \widetilde{\mathcal{M}} : \tau(\Phi(|f|) \leq 1\}$$

- He showed that this is a linear space which becomes a Banach space when equipped with the Luxemburg-Nakano norm

$$\|f\|_\Phi = \inf\{\lambda > 0 : \tau(\Phi(|f|/\lambda)) \leq 1\}.$$

- One has

$$L_{\Phi}^{ncO} = \{f \in \widetilde{\mathcal{M}} : \tau(\Phi(\lambda|f|)) < \infty \quad \text{for some} \quad \lambda = \lambda(f) > 0\}.$$

Xu; Doods, Dodds, de Pagter approach

- $f \in \widetilde{\mathcal{M}}$ and $t \in [0, \infty)$, the generalized singular value $\mu_t(f)$ is defined by $\mu_t(f) = \inf\{s \geq 0 : \tau(1\!\!1 - e_s(|f|)) \leq t\}$ where $e_s(|f|)$ $s \in \mathbb{R}$ is the spectral resolution of $|f|$.

- The function $t \to \mu_t(f)$ will generally be denoted by $\mu(f)$.

- Banach Function Space of measurable functions on $(0, \infty)$.

- A function norm ρ on $L^0(0, \infty)$ is defined to be a mapping $\rho : L^0_+ \to [0, \infty]$ satisfying
  - $\rho(f) = 0$ iff $f = 0$ a.e.
  - $\rho(\lambda f) = \lambda\rho(f)$ for all $f \in L^0_+, \lambda > 0$.
  - $\rho(f + g) \leq \rho(f) + \rho(g)$ for all .
  - $f \leq g$ implies $\rho(f) \leq \rho(g)$ for all $f, g \in L^0_+$.

- Such a ρ may be extended to all of L^0 by setting $\rho(f) = \rho(|f|)$.

- Define $L^\rho(0, \infty) = \{f \in L^0(0, \infty) : \rho(f) < \infty\}$. If now $L^\rho(0, \infty)$ turns out to be a Banach space when equipped with the norm $\rho(\cdot)$, we refer to it as a Banach Function space.

- If $\rho(f) \leq \liminf_n (f_n)$ whenever $(f_n) \subset L^0$ converges almost everywhere to $f \in L^0$, we say that ρ has the Fatou Property.

- If this implication only holds for $(f_n) \cup \{f\} \subset L^\rho$, we say that ρ is lower semi-continuous.

- If $f \in L^\rho$, $g \in L^0$ and $\mu_t(f) = \mu_t(g)$ for all $t > 0$, forces $g \in L^\rho$ and $\rho(g) = \rho(f)$, we call L^ρ rearrangement invariant (or symmetric).

- Dodds, Dodds and de Pagter formally defined the noncommutative space $L^\rho(\widetilde{\mathcal{M}})$ to be

$$L^\rho(\widetilde{\mathcal{M}}) = \{f \in \widetilde{\mathcal{M}} : \mu(f) \in L^\rho(0,\infty)\}$$

  and showed that if ρ is lower semicontinuous and $L^\rho(0,\infty)$ rearrangement-invariant, $L^\rho(\widetilde{\mathcal{M}})$ is a Banach space when equipped with the norm $\|f\|_\rho = \rho(\mu(f))$.

- For any Young's function Φ, the Orlicz space $L^\Phi(0,\infty)$ is known to be a rearrangement invariant Banach Function space with the norm having the Fatou Property.

- Thus taking ρ to be $\|\cdot\|_\Phi$, the very general framework of Dodds, Dodds and de Pagter presents us with an alternative approach to realising noncommutative Orlicz spaces.

- Non-commutative regular systems

- 1. $n_\tau = \{x \in \mathcal{M} : \tau(x^* x) < +\infty\}$.
  2. (*definition ideal of the trace* τ) $m_\tau = \{xy : x, y \in n_\tau\}$.
  3. $\omega_x(y) = \tau(xy), \quad x \geq 0$.

- 1. if $x \in m_\tau$, and $x \geq 0$, then $\omega_x \in \mathcal{M}_*^+$.
  2. If $L^1(\mathcal{M}, \tau)$ stands for the completion of $(m_\tau, ||\cdot||_1)$ then $L^1(\mathcal{M}, \tau)$ is isometrically isomorphic to $\mathcal{M}_*$.
  3. $\mathcal{M}_{*,0} \equiv \{\omega_x : x \in m_\tau\}$ is norm dense in $\mathcal{M}_*$.

  Finally, denote by $\mathcal{M}_*^{+,1}$ ($\mathcal{M}_{*,0}^{+,1}$) the set of all normalized normal positive functionals in $\mathcal{M}_*$ (in $\mathcal{M}_{*,0}$ respectively).

- **Definition 16.** *The noncommutative statistical model consists of a quantum measure space* $(\mathcal{M}, \tau)$*, "quantum densities with respect to* τ*" in the form of* $\mathcal{M}_{*,0}^{+,1}$*, and the set of* τ*-measurable operators* $\widetilde{\mathcal{M}}$*.*

- **Definition 17.**

$$L_x^{quant} = \{g \in \widetilde{\mathcal{M}} : \quad 0 \in D(\widehat{\mu_x^g(t)})^0, \quad x \in m_\tau^+\}, \tag{15}$$

where $D(\cdot)^0$ *stands for the interior of the domain* $D(\cdot)$ *and*

$$\widehat{\mu_x^g(t)} = \int \exp(t\mu_s(g))\mu_s(x)ds, \qquad t \in \mathbb{R}. \tag{16}$$

(Notice that the requirement that $0 \in D(\widehat{\mu_x^g(t)})^0$*, presupposes that the transform* $\widehat{\mu_x^g(t)}$ *is well-defined in a neighborhood of the origin.)*

- We remind that above and in the sequel $\mu(g)$ ($\mu(x)$) stands for the function $[0,\infty) \ni t \mapsto \mu_t(g) \in [0,\infty]$ ($[0,\infty) \ni t \mapsto \mu_t(x) \in [0,\infty]$ respectively).

- To give a non-commutative generalization of Pistone-Sempi theorem we need a generalization of Dodds, Dodds, de Pagter approach i.e. that one which was presented in Section 4.

- **Definition 18.** *Let $x \in L^1_+(\mathcal{M},\tau)$ and let ρ be a Banach function norm on $L^0((0,\infty),\mu_t(x)dt)$. In the spirit of Dodds, Dodds, de Pagter, we then formally define the weighted noncommutative Banach function space $L^\rho_x(\widetilde{\mathcal{M}})$ to be the collection of all $f \in \widetilde{\mathcal{M}}$ for which $\mu(f)$ belongs to $L^\rho((0,\infty),\mu_t(x)dt)$. For any such f we write $\|f\|_\rho = \rho(\mu(f))$.*

- **Remark 19.** *Comparing commutative and non-commutative regular statistical models, we note that $\mu_t(x)$ (the Lebesgue measure dt) in Definition 18 stands for f ($d\nu$, respectively).*

- The mentioned generalization of Dodds, Dodds, de Pagter approach is contained in:

Theorem 20. *Let* $x \in L^1_+(\mathcal{M}, \tau)$. *Let* ρ *be a rearrangement-invariant Banach function norm on* $L^0((0,\infty), \mu_t(x)dt)$ *which satisfies the Fatou property,* $\rho(\chi_E) < \infty$ *and* $\int_E f d\mu \leq C_E \rho(f)$ *for* $E : \mu(E) < \infty$. *Then* $L^\rho_x(\widetilde{\mathcal{M}})$ *is a linear space and* $\| \cdot \|_\rho$ *a norm. Equipped with the norm* $\| \cdot \|_\rho$, $L^\rho_x(\widetilde{\mathcal{M}})$ *is a Banach space which injects continuously into* $\widetilde{\mathcal{M}}$.

- and the generalization of Pistone-Sempi is given by

Theorem 21. *The set* L^{quant}_x *coincides with the closed subspace of the weighted Orlicz space* $L^{\cosh -1}_x(\widetilde{\mathcal{M}}) \equiv L^{\Psi}_x(\widetilde{\mathcal{M}})$ *(where* $\Psi = \cosh - 1$*) of noncommutative random variables with a fixed expectation.*

- To show that statistics and thermodynamics can be well established for noncommutative regular statistical systems, we note that

Proposition 22. *Let $\mathcal{M}$ be a semifinite von Neumann algebra with an fns trace τ. By χ_I will denote spectral projections of f. Then*

$$f \in L_{\log L}(\widetilde{M})^+ \quad \textit{with} \quad \tau(\chi_{[0,1]}) < \infty \Longleftrightarrow \quad \tau(|f \log(f)|) \quad \textit{exists}$$

- Consequently, if the "state" is taken from the noncommutative Zygmund space $L_{\log L}(\widetilde{M})$, then the entropy function exists!

  (Note that the above Proposition describes the quantum counterpart of finite measure case!)

- In conclusion, analogously to the commutative case, we got

Corollary 23.

$$< L^{cosh-1}, L^{(\cdot)ln(\cdot+\sqrt{1+(\cdot)^2})-\sqrt{1+(\cdot)^2}+1} >$$

or equivalently (for "finite measure" case)

$$< L_{exp}, LlogL >$$

provides the proper framework for the description of non-commutative regular statistical systems, where now Orlicz (and Zygmund) spaces are noncommutative. Note, a general case needs some modifications.