Riesz transforms and Hodge decomposition on complete Riemannian manifolds

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Outline

What are Riesz transforms

Riesz transforms are basic examples of singular integral operators and play important role in analysis and geometry.

Formally, the Riesz transform on a Riemannian manifold is written as

$$R = \nabla(-\Delta)^{-1/2}.$$

The study of Riesz transforms dates back to the beginning of the last century (Fatou, ..., Kolmogorov, M. Riesz).

History: Fatou's thesis

Let $f \in \mathrm{C}(\mathbb{S}^1, \mathbb{R})$, u be its harmonic extension in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. By the Poisson formula, we have

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{S^1} \frac{1 - r^2}{1 - 2r\cos(\theta - \xi) + r^2} f(e^{i\xi}) d\xi.$$

The conjugate function of u, denoted by v, defined as the solution of the Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

is uniquely determined by f (up to an additional constant).

Hilbert transform: Fatou's thesis

The conjugate Poisson kernel representation formula holds

$$v(re^{i\theta}) = \frac{1}{2\pi} \int_{S^1} \frac{\sin(\theta - \xi)}{1 - 2r\cos(\theta - \xi) + r^2} f(e^{i\xi}) d\xi.$$

In his celebrated thesis, Fatou proved that the non-tangential limit of v, denoted by

$$Hf(e^{i\theta}) := \lim_{r \to 1-} v(re^{i\theta})$$

exists almost surely on $\mathbb{S}^1$. Moreover, one has

$$Hf(e^{i\theta}) = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{|\theta - \xi| \geq \varepsilon} \frac{f(e^{i\xi})}{\tan \frac{\theta - \xi}{2}} d\xi, \quad a.s.\ \theta \in [0, 2\pi].$$

Hilbert transform

- The Hilbert transform on the unit circle S^1:

$$\mathrm{H}f(e^{i\theta}) = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{|\theta-\xi|\geq\varepsilon} \frac{f(e^{i\xi})}{\tan\frac{\theta-\xi}{2}} d\xi, \quad a.s. \;\; \theta \in [0, 2\pi].$$

- Let

$$f(e^{i\theta}) = a_0 + \sum_{n=1}^{\infty} a_n \sin(n\theta) + b_n \cos(n\theta).$$

Then

$$\mathrm{H}f(e^{i\theta}) = \sum_{n=1}^{\infty} b_n \sin(n\theta) - a_n \cos(n\theta).$$

- By the Plancherel formula, we have

$$\|\mathrm{H}f\|_2^2 = \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \|f - \pi(f)\|_2^2,$$

where

$$\pi(f) = a_0 := \int_{S^1} f(e^{i\theta}) \frac{d\theta}{2\pi}.$$

Hilbert transform on $\mathbb{R}$

The Hilbert transform on the real line $\mathbb{R}$ is defined by

$$\mathrm{H}f(x) := \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} \frac{f(y)}{x-y} dy, \quad a.s.\ x \in \mathbb{R}.$$

In Fourier analysis, for all $f \in L^2(\mathbb{R})$, we have

$$\widehat{\mathrm{H}f}(\xi) = i \frac{\xi}{|\xi|} \widehat{f}(\xi), \quad \forall \xi \in \mathbb{R}.$$

By the Plancherel formula, we then obtain

$$\|\mathrm{H}f\|_2 = \|f\|_2, \quad \forall f \in L^2(\mathbb{R}).$$

Hence, the Hilbert transform H is an isometry in L^2.

Landmark papers of A. N. Kolmogorov and M. Riesz

A. N. Kolmogorov: *Sur les fonctions harmoniques conjuguées et les séries de Fourier*. Fund. Math. 7 (1925), 24-29.

M. Riesz: *Sur les fonctions conjuguées*. Math. Zeit. 27 (1927), 218-244

Theorem of A. N. Kolmogorov (1925). There exists a constant $C > 0$ such that for all $\lambda > 0$, and for all $f \in L^1(\mathbb{R}, dx)$,

$$\mathrm{Leb}(\{x \in \mathbb{R} : |\mathrm{H}f(x)| \geq \lambda\}) \leq \frac{C\|f\|_1}{\lambda}.$$

Theorem of M. Riesz (1927). For all $p > 1$, there exists a constant $C_p > 0$ such that for all $f \in L^p(\mathbb{R}, dx)$,

$$\|\mathrm{H}f\|_p \leq C_p\|f\|_p.$$

Remark

1. The proof of M. Riesz's theorem uses the method of complex analysis.

2. The truncated Hilbert transform

$$\mathrm{H}_\varepsilon f(x) := \frac{1}{2\pi} \int_{|x-y|\geq\varepsilon} \frac{f(y)}{x-y} dy, \quad x \in \mathbb{R}$$

has been used by Carleson (Acta Math 1966) to solve the Luzin conjecture.

Riesz transforms on $\mathbb{R}^n$

To extend the L^p-continuity of the Hilbert transform to higher dimensional Euclidean spaces, Caldéron and Zygmund (Acta Math 1952) developed the real method in the study of singular integral operators on Euclidean spaces.

Riesz transform has been put as the corner stone in the Calderon-Zygmund theory.

Riesz transforms on $\mathbb{R}^n$

Definition

For $f \in C_0^\infty(\mathbb{R}^n)$, $j = 1, \ldots, n$, the j-th Riesz transform $R_j f$ of f is defined by

$$R_j f(x) := \frac{\Gamma((n+1)/2)}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} f(y) \frac{x_j - y_j}{|x-y|^{n+1}} dy, \quad a.s.\ x \in \mathbb{R}^n.$$

Formally, we have

$$R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2}.$$

Theorem (Calderon-Zygmund1952, cf. E.-M. Stein1970)

For all $p > 1$, there exists a constant $C_p > 0$ such that

$$\|R_j f\|_p \leq C_p \|f\|_p, \quad \forall f \in L^p(\mathbb{R}^n).$$

Riesz transforms on Lie groups

In 1970, E.-M. Stein developed the Littlewood-Paley theory for symmetric sub-Markovian semigroups on a measurable spaces.

Using the Littlewood-Paley inequalities, Stein proved the L^p-boundedness of the Riesz transforms on compact or compact type Lie groups.

Riesz transforms on Lie groups

Let G be a compact or compact type Lie group, e the unit element, $e_1, \dots, e_n$ an orthonormal basis of the Lie algebra $\mathcal{G} = T_e G$.
Let $X_1, \dots, X_n$ be the left invariant vector fields on G defined by

$$X_i(g) = (L_g)_*(e_j), \quad \forall g \in G,\ j = 1, \dots, n.$$

The Casimir operator on G is defined by

$$\Delta = X_1^2 + \dots + X_n^2.$$

The Riesz transforms on G are defined by

$$R_j = X_j(-\Delta)^{-1/2}, \quad j = 1, \dots, n.$$

Riesz transforms on Lie groups

Theorem (E.-M. Stein 1970)

Let G be a compact or compact type Lie group. Then for all $p > 1$, there exists a constant $C_p > 0$ such that

$$\|R_j f\|_p \leq C_p \|f\|_p, \quad \forall f \in L^p(G, dx),$$

where dx denotes the Haar measure on G.

Riesz transform on Gaussian spaces

Let $L = \Delta - x \cdot \nabla$ be the Ornstein-Uhlenbeck operator on the Gaussian space $(\mathbb{R}^n, d\gamma_n)$, where

$$d\gamma_n(x) = \frac{e^{-\frac{\|x\|^2}{2}}}{(2\pi)^{n/2}} dx.$$

Following P.-A. Meyer, the Riesz transform associated to the Ornstein-Uhlenbeck operator is defined by

$$\nabla(-L)^{-1/2} f = \frac{1}{\sqrt{\pi}} \int_0^\infty \nabla e^{tL} f \frac{dt}{\sqrt{t}},$$

where $f \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\gamma_n(f) = \int_{\mathbb{R}^{\ltimes}} f(x) d\gamma_n(x) = 0.$$

P.-A. Meyer's inequality

Using a probabilistic approach to the Littlewood-Paley-Stein inequalities, P.-A. Meyer proved the following remarkable

Theorem (P.-A. Meyer 1982)

For all $p > 1$, there exists a constant C_p which is independent of $n = \mathrm{dim}\mathbb{R}^n$ such that

$$\|\nabla(-L)^{-1/2}f\|_p \leq C_p\|f\|_p, \quad \forall\, f \in L^p(\mathbb{R}^n, \gamma_n).$$

Consequently, it holds that

$$C_p^{-1}(\|f\|_p + \|\nabla f\|_p) \leq \|(1-L)^{1/2}f\|_p \leq C_p(\|f\|_p + \|\nabla f\|_p).$$

Moreover, the above inequalities remain true on the infinite dimensional Wiener space equipped with the Wiener measure.

Using the P.-A. Meyer's inequalities, Airault-Malliavin and Sugita proved the following beautiful result which consists of the base of the quasi-sure analysis on infinite dimensional Wiener space (Malliavin 82/97, Ren 90, Bouleau-Hirsch 91, etc.).

Theorem (Airault-Malliavin 1990, H. Sugita 1990)

All the positive distributions (in the sense of Watanabe) on the Wiener space are probability measures.

Analytic proof of P.A. Meyer's inequality

In 1986, G. Pisier gave a simple analytic proof of P.-A. Meyer's inequality without the Littlwood-Paley-Stein theory, but using the rotational invariance of the Gaussian measure.

Pisier's method has been extended by Malliavin and Nualart to Riesz transforms associated with the Ornstein-Uhlenbeck operator acting on functionals defined on Wiener spaces and with values in UMD Banach spaces, an area which has increasing interest in the study of geometry on Banach spaces.

Riesz transforms on Riemannian manifolds

Let (M, g) be a complete Riemannian manifold, ∇ the gradient, and Δ the Laplace-Beltrami on (M, g).

Definition (Stein70, Strichartz83, Lohoué85)

- *The Riesz potential on (M, g) is defined by*

$$(-\Delta)^{-1/2} f = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{t\Delta} f \frac{dt}{\sqrt{t}}.$$

- *The Riesz transform on (M, g) is defined by*

$$\nabla(-\Delta)^{-1/2} f = \frac{1}{\sqrt{\pi}} \int_0^\infty \nabla e^{t\Delta} f \frac{dt}{\sqrt{t}}.$$

Difficulty: Singularity appears when $t \to 0$ and $t \to \infty$. The problem of existence of the limit (in which sense) of these singular integrals should be precisely studied and determined.

Neumann to Dirichlet operator

Let $\mathcal{M} = M \times \mathbb{R}^+$, on which equipped with the product Riemannian metric, i.e.,

$$ds^2_{\mathcal{M}}(x, y) = ds^2_M(x) + dy^2.$$

Let Δ_M be the Laplace-Beltrami operator on M. Then

$$\Delta_{\mathcal{M}} = \Delta_M + \frac{\partial^2}{\partial y^2}.$$

Consider the Neumann boundary problem on $\mathcal{M} = M \times \mathbb{R}$:

$$\begin{aligned} \Delta_{\mathcal{M}} u &= 0 \ \text{ in } \ \mathcal{M}, \\ \frac{\partial}{\partial n} u &= f \ \text{ on } \ M. \end{aligned}$$

Suppose that $u(x, y)$ is given by the Poisson integral of the Dirichlet boundary function $\widetilde{f} := \lim_{y \to 0} u(x, y)$. Then

$$u(x, y) = e^{-y\sqrt{-\Delta_M}} \widetilde{f}(x), \quad (x, y) \in M \times \mathbb{R}^+.$$

Neumann to Dirichlet operator

The Dirichlet and the Neumann boundary conditions are related by

$$f(x) \quad = \quad -\frac{\partial}{\partial y} e^{-y\sqrt{-\Delta_M}} f(x,y)\Big|_{y=0} = \sqrt{-\Delta_M}\widetilde{f}(x).$$

In other words, the Neumann to Dirichlet operator is defined by

$$\widetilde{f} = (-\Delta_M)^{-1/2} f.$$

Hence, at least formally, the Riesz transform of f on (M, g), defined by

$$Rf := \nabla(-\Delta_M)^{-1/2} f = \nabla \lim_{y \to 0} u(\cdot, y),$$

is the gradient of the non-tangential limit of the solution of the Neumann boundary problem.

Riesz transforms are pseudo-differential operators

- The Riesz potential $(-\Delta)^{-1/2}$ is a pseudo-differential operator of order -1.
- The Riesz transform $\nabla(-\Delta)^{-1/2}$ is a pseudo-differential operator of order zero.

By the theory of pseudo-differential operators (Seely), the Riesz transform is always bounded in L^p for all $p \in (1, \infty)$ on all compact Riemannian manifolds.

The L^2-continuity of Riesz transforms

By the Gaffney L^2-integration by parts formula, we have

Proposition (Strichartz JFA1983)

Let (M, g) be a complete Riemannian manifold. Then, for all $f \in C_0^\infty(M)$,

$$\|\nabla f\|_2^2 = -\langle\langle \Delta f, f\rangle\rangle = \|(-\Delta)^{1/2} f\|_2^2.$$

Hence, for all $f \in C_0^\infty(M)$ with $Hf = 0$, we have

$$\|\nabla(-\Delta)^{-1/2} f\|_2 = \|f\|_2,$$

where

$$H : \quad L^2(M) \longrightarrow \mathrm{Ker}(\Delta) \cap L^2(M)$$

is the harmonic projection. By continuity extension, $\nabla(-\Delta)^{1/2}$ extends as an L^2-isometry from $L^2(M) \setminus \mathrm{Ker}\Delta$ to $L^2(M, TM)$.

L^p-continuity of Riesz transforms

Hence, on any complete Riemannian manifold (M, g), the Riesz transform $R = \nabla(-\Delta)^{-1/2}$ is an isometry in L^2 :

$$\|Rf\|_2 = \|f\|_2.$$

A fundamental problem in harmonic analysis is the following

Problem (E.-M. Stein1970, Strichartz1983, Lohoue1985)

Under which condition on a complete non-compact Riemannian manifold, the Riesz transform is bounded in L^p for all (or some) $p > 1$ (but $p \neq 2$), i.e., $\exists\ C_p > 0$ such that

$$\|Rf\|_p \leq C_p \|f|_p, \quad \forall\ f \in L^p(M)?$$

Main methods

The main methods developed in the past decades are

- the classical method in harmonic and geometric analysis
- the Littlewood-Paley-Stein theory for Riesz transforms
  - analytic approach based on Calderon-Zygmund theory
  - probabilistic approach based on martingale inequalities

Classical method in harmonic and geometric analysis

To prove the L^p-boundedness of the Riesz transform

$$\nabla(-\Delta)^{-1/2}f(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty \int_M f(y)\frac{\nabla_x p_t(x,y)}{\sqrt{t}}dydt,$$

one needs to use

- the double volume property for the Calderon-Zygmund decomposition on manifolds
- the gradient and Hessian estimate of the heat kernel on manifolds (Li-Yau, Dodziuk, Donnelly-Li, Schoen, Hamilton, Sheu, Hsu, Stroock...)

This method does not work on Gaussian spaces equipped with the Wiener measure.

Littlewood-Paley-Stein theory for Riesz transforms

The idea is to prove two Littlewood-Paley-Stein inequalities:

- the L^p-boundedness of the horizontal LPS function g_2 acting on scalar functions on Riemannian manifolds

$$g_2(f)(x) = \left(\int_0^\infty t \left| \nabla e^{-t\sqrt{a-\Delta}} f(x) \right|^2 dt \right)^{1/2},$$

- the L^q-boundedness of the vertical LPS function $g_1(\omega)$ acting on differential one-forms on Riemannian manifolds

$$g_1(\omega)(x) = \left(\int_0^\infty t \left| \frac{\partial}{\partial t} e^{-t\sqrt{a+\Box}} \omega(x) \right| dt \right)^{1/2}.$$

where $q = \frac{p}{p-1}$, $\omega \in C_0^\infty(M, \Lambda T^* M)$ is one-form,

- Key tool: the Bochner-Weitzenböck formula

$$\Box = -\Delta + Ric.$$

One of the main ingredients in the LPS theory is the commutation formula

$$de^{-t\sqrt{a-\Delta}}f = e^{-t\sqrt{a+\Box}}(df),$$

where $\Box$ is the Hodge Laplacian on one-forms.
By the spectral decomposition and the Littlewood-Paley identity, it holds that (Bakry 1984, J. Chen 1985)

$$\langle \nabla(a-\Delta)^{-1/2}f, \omega\rangle_{L^2} \leq 4\|g_2(f)\|_p\|g_1(\omega)\|_q.$$

Therefore, if the following LPS inequalities holds

$$\|g_2(f)\|_p \leq A_p\|f\|_p, \quad \|g_1(\omega)\|_q \leq B_q\|\omega\|_q,$$

then by duality argument we have

$$\|\nabla(a-\Delta)^{-1/2}f\|_p \leq 4A_pB_q\|f\|_p.$$

Some known results

- Non-compact Rienannian symmetric space of rank 1
  Strichartz (JFA83)
- Riemannian manifolds with bounded geometry condition and strictly positive bottom of L^2-spectrum of Laplacian
  Lohoué (JFA85, CRAS85, CRAS90, Orsay Preprint94, MathNachr06)
- Riemannian manifolds with Ricci curvature non-negative or bounded from below
  Bakry (87, 89), C.-J. Chen (88), Chen-Luo (88), J.-Y. Li (91), Shigekawa-Yoshida (92), Yoshida (92, 94)
- Counter-examples: Lohoué(94), Coulhon-Ledoux (94), Coulhon-Duong (99), Carron-Coulhon-Hassell (DMJ06).

Theorem (Bakry 1987)

Let (M, g) be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exists a constant $a \geq 0$ such that

$$Ric + \mathrm{Hess}\phi \geq -a.$$

Then, $\forall p \in (1, \infty)$, $\exists C_p > 0$ such that

$$\|\nabla(a - L)^{-1/2} f\|_p \leq C_p \|f\|_p, \quad \forall f \in C_0^\infty(M),$$

where

$$\|f\|_p^p := \int_M |f(x)|^p e^{-\phi} dv.$$

Ricci curvature

By the Gauss lemma, in the geodesic normal coordinates near any point p in a Riemannian manifold (M, g), we have

$$g_{ij} = \delta_{ij} + O(|x|^2).$$

In these coordinates, the metric volume form then has the following Taylor expansion at p:

$$d\nu_g = \left(1 - \frac{1}{6} R_{ij} x_i x_j + O(|x|^2)\right) d\nu_{\text{Euclidean}}.$$

Theorem (S.-T. Yau 1975)

Let M be a complete Riemannian manifold with $Ric \geq K$. Then, for any non-negative harmonic function on M, i.e., $\Delta u = 0$, we have

$$|\nabla u| \leq \sqrt{(n-1)K} u.$$

Some new results

Some new results have been established in recent years in the following setting

- Riemaniann manifolds satisfying the doubling volume property, relative Faber-Krahn inequalities and an additional heat kernel regularity conditions
  Coulhon-Duong (TAMS99, CRAS01, CPAM03),
  Auscher-Couhon-Duong-Hofmann (ASENS04),
  Auscher-Coulhon (ASNSP05),
  Carron-Coulhon-Hassell (DMJ2006).

- Riemannian manifolds on which the negative part of Ricci curvature satisfies $L^{n/2+\varepsilon}$-integrability conditions
  X.-D. Li (Revista Mat. Iberoamericana 2006)

Theorem (Li RMI2006)

Let (M, g) *be an n-dimensional Cartan-Hadamard manifold. Suppose*
$(C1)$ *there exist a constant* $C > 0$ *and a fixed point* $o \in M$ *such that*

$$Ric(x) \geq -C[1 + d^2(o, x)], \quad \forall x \in M,$$

$(C2)$ *there exist some constants* $c \geq 0$ *and* $\epsilon > 0$ *such that*

$$(K + c)^- \in L^{\frac{n}{2}+\epsilon}(M),$$

where

$$K(x) = \inf\{\langle Ric(x)v, v\rangle : v \in T_xM,\ |v| = 1\}, \quad \forall x \in M.$$

Then, for all $p \geq 2$*, the Riesz transform* $\nabla(-\Delta)^{-1/2}$ *is bounded in* L^p*.*

Theorem (Li RMI2006)

Let M be a complete Riemannian manifold, and $\phi \in C^2(M)$. Suppose that there exists a constant $m > 2$ such that the Sobolev inequality holds

$$\|f\|_{\frac{2m}{m-2}} \leq C_m(\|\nabla f\|_2 + \|f\|_2), \quad \forall f \in C_0^\infty(M).$$

Suppose that there exist some constants $c \geq 0$ and $\epsilon > 0$ such that

$$(K + c)^- \in L^{\frac{m}{2}+\epsilon}(M, e^{-\phi}dv),$$

where

$$K(x) = \inf\{\langle(Ric + \mathrm{Hess}\phi)v, v\rangle : \ v \in T_xM, \ |v| = 1\}.$$

Then, the Riesz transform $\nabla(a - L)^{-1/2}$ is bounded in $L^p(M, e^{-\phi}dv)$ for all $p \geq 2$ and for all $a > 0$.

Sharp L^p-norm estimates of Riesz transforms

In 1972, Pichorides proved that the L^p-norm of the Hilber transform is given by

$$\|H\|_{p,p} = \cot\left(\frac{\pi}{2p^*}\right), \quad \forall p > 1,$$

where

$$p^* = \max\left\{p, \frac{p}{p-1}\right\}.$$

In 1996, Iwaniec and Martin (Crelles1996) proved that

$$\|R_j\|_{p,p} = \cot\left(\frac{\pi}{2p^*}\right), \quad \forall p > 1.$$

Sharp L^p-norm estimates of Riesz transforms

In 1983, E.-M. Stein (Bull. AMS 1983) proved that the best constant in the L^p-continuity inequality of the Riesz transform

$$R = \nabla(-\Delta)^{-1/2}$$

on $\mathbb{R}^n$ is independent of $n = \dim \mathbb{R}^n$. More precisely,

Theorem (E.-M. Stein 1983)

For all $p > 1$, there exists a constant $C_p > 0$ depending only on p and independent of $n = \dim \mathbb{R}^n$ such that

$$\|Rf\|_p \leq C_p \|f\|_p, \quad \forall f \in L^p(\mathbb{R}^n, dx),$$

where

$$\|Rf\|_p := \left\| \left(\sum_{j=1}^{n} |R_j f|^2 \right)^{1/2} \right\|_p .$$

Bañuelos-Wang's results on Riesz transforms

In 1995, using the Gundy-Varopoulos representation formula of Riesz transforms, and the sharp L^p-Burkholder inequality for martingale transforms, Bañuelos and Wang (Duke Math. J. 1995) gave a probabilistic proof of Iwaniec-Martin's result.

Moreover, they proved the following

Theorem (Bañuelos-Wang 1995)

For all $p > 1$ and $f \in L^p(\mathbb{R}^n, dx)$, it holds

$$\|Rf\|_p \leq 2(p^* - 1)\|f\|_p.$$

In other words,

$$\|R\|_{p,p} := \sup_{f \in L^p(\mathbb{R}^n, dx)} \frac{\|Rf\|_p}{\|f\|_p} \leq 2(p^* - 1).$$

Gundy-Varopoulos representation formula

Let X_t be Brownian motion on $\mathbb{R}^n$, B_t Brownian motion on $\mathbb{R}$, independent to X_t, $B_0 = y$, with infinitesimal generator $\frac{d^2}{dy^2}$. (X_t, B_t) is called the background radiation process. Let

$$\tau = \inf\{t > 0 : B_t = 0\}.$$

Theorem (Gundy-Varopoulos79, Gundy-Silverstein82)

Let $f \in C_0^\infty(\mathbb{R}^n)$, and let $u(x, y)$ be the Poisson integral of f:

$$u(x, y) := e^{-y\sqrt{-\Delta}} f(x), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Then

$$-\frac{1}{2} Rf(x) = \lim_{y\to\infty} E_y \left[\int_0^\tau \nabla_x u(X_s, B_s) dB_s \,\middle|\, X_\tau = x \right],$$

where

$$\nabla = (\nabla_x, \frac{\partial}{\partial y}) = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y}).$$

Gundy-Varopoulos formulas on Riemiann manifolds

Notations

- (M, g) a complete Riemannian manifold, $\phi \in C^2(M)$,

$$L = \Delta - \nabla\phi \cdot \nabla, \quad d\mu = e^{-\phi}dv,$$

- X_t: L-diffusion on M with initial distribution μ,
- B_t: Brownian motion on $\mathbb{R}$, independent of X_t,

$$B_0 = y > 0, \text{ and } \quad E[B_t^2] = 2t.$$

- (X_t, B_t) is called the background radiation process (following Gundy-Varopolous) .

- $$\tau = \inf\{t > 0 : B_t = 0\}.$$
- Let $M_t \in \mathrm{End}(T^*_{X_0}M, T^*_{X_t}M)$ be the solution to the following covariant SDE along the trajectory of L-diffusion X_t

$$\begin{aligned} \nabla_{\circ dX_t} M_t &= -Ric(L)(X_t)M_t, \\ M_0 &= \mathrm{Id}_{T_{X_0}M}, \end{aligned}$$

where

$$Ric(L) = Ric + \nabla^2\phi,$$

and $\nabla_{\circ dX_t}$ denotes the Itô stochastic covariant derivative wrt the Levi-Civita connection on M along $\{X_s, s \in [0, t]\}$.

Duality between BM starting from ∞ and Bessel 3

In his paper Le dual de $H^1(\mathbb{R}^n)$ (LNM581, 1977), P.-A. Meyer described the duality between the Brownian motion starting from infinity and Bessel 3 as follows :

D'une manière intuitive, on peut donc dire que le retourné du processus de Bessel issu de λ_0 est le "mouvement brownien venant de l'infini et tué en 0", où $\lambda_0 = dx \otimes \delta_0$.

In an intuitive way, one can say that the time-reversal process of the Bessel process starting from λ_0 is the "Brownian motion starting from infinity and killed at 0", where $\lambda_0 = dx \otimes \delta_0$.

Gundy-Varopoulos formula on Riemannian manifolds

Theorem (Li PTRF 2008, Preprint2011)

Suppose that $Ric + \nabla^2\phi \geq -a$, *where* $a \geq 0$ *is a constant. Let*

$$u_a(x,y) = e^{-y\sqrt{a-L}} f(x), \quad x \in M, y > 0.$$

Then

$$\begin{aligned} & -\frac{1}{2}\nabla(a-L)^{-1/2} f(x) \\ = & \lim_{y\to+\infty} E_y\left[e^{-a\tau} M_\tau \int_0^\tau e^{as} M_s^{-1} \nabla u_a(X_s, B_s) dB_s \,|\, X_\tau = x\right]. \end{aligned}$$

Remark. In particular, if $Ric + \nabla^2\phi = -k$, then

$$M_t = e^{kt} U_t.$$

Example 1: Riesz transforms on $\mathbb{R}^n$

Let $M = \mathbb{R}^n, \ \phi = 0$. Then

$$L = \Delta, \quad Ric + \nabla^2\phi = 0.$$

Thus, we can recapture the Gundy-Varopoulos representation formula of Riesz transforms on $\mathbb{R}^n$:

$$-\frac{1}{2}\nabla(-\Delta)^{-1/2}f(x) = \lim_{y\to+\infty} E_y\left[\int_0^\tau \nabla e^{-B_s\sqrt{-\Delta}} f(X_s)dB_s \,|\, X_\tau = x\right].$$

Example 2: Riesz transforms on Gaussian spaces

Let $M = \mathbb{R}^n$, $\phi = \frac{\|x\|^2}{2} + \frac{n}{2}\log(2\pi)$. Then

$$L = \Delta - x \cdot \nabla, \quad Ric + \nabla^2\phi = \mathrm{Id}.$$

We recapture Gundy's representation formula (1986) for P.-A. Meyer's Riesz transform on n-dimensional Gaussian space:

$$-\frac{1}{2}\nabla(a-L)^{-1/2}f(x)$$
$$= \lim_{y\to+\infty} E_y\left[e^{-(a+1)\tau}\int_0^\tau e^{(a+1)s}\nabla e^{-B_s\sqrt{a-L}}f(X_s)dB_s \mid X_\tau = x\right].$$

In particular, when $a = 0$, we get

$$-\frac{1}{2}\nabla(-L)^{-1/2}f(x)$$
$$= \lim_{y\to+\infty} E_y\left[e^{-\tau}\int_0^\tau e^{s}\nabla e^{-B_s\sqrt{-\Delta}}f(X_s)dB_s \mid X_\tau = x\right].$$

Example 3: Riesz transforms on Spheres

- Let $M = S^n$ and $\phi = 0$. Then $Ric = n - 1$. We recapture Arcozzi's representation formula (1998)

$$
\begin{aligned}
& -\frac{1}{2}\nabla(a - \Delta_{S^n})^{-1/2} f(x) \\
= & \lim_{y \to \infty} E_y \left[e^{-(a+n-1)\tau} \int_0^\tau e^{(a+n-1)s} \nabla e^{-B_s \sqrt{a - \Delta_{S^n}}} f(X_s) dB_s \mid X_\tau = x \right]
\end{aligned}
$$

- Let $M = S^n(\sqrt{n-1})$ and $\phi = 0$. Then $Ric = 1$. This yields

$$
\begin{aligned}
& -\frac{1}{2}\nabla(a - \Delta_M)^{-1/2} f(x) \\
= & \lim_{y \to +\infty} E_y \left[e^{-(a+1)\tau} \int_0^\tau e^{(a+1)s} \nabla e^{-B_s \sqrt{a - \Delta_M}} f(X_s) dB_s \mid X_\tau = x \right].
\end{aligned}
$$

Example 4: Riesz transform on Wiener space

Taking $n \to \infty$ in the above formula, and using the Poincaré limit, we can obtain Gundy's representation formula on Wiener space (Gundy 1986, S. Song 1992)

$$
\begin{aligned}
&-\frac{1}{2}\nabla(a-L)^{-1/2}f(x) \\
&= \lim_{y\to+\infty} E_y\left[e^{-(a+1)\tau}\int_0^\tau e^{(a+1)s)}\nabla e^{-B_s\sqrt{a-L}}f(X_s)dB_s \mid X_\tau = x\right],
\end{aligned}
$$

where

- L denotes the Ornstein-Uhlenbeck operator on Wiener space,
- X_t denotes the Ornstein-Uhlenbeck process on Wiener spacee.

Sharp L^p-norm estimates of Riesz transforms on Riemannian manifolds

Theorem (Li PTRF2008, Preprint2011)

Suppose that $Ric(L) = Ric + \nabla^2\phi = 0$. *Then, for all* $p > 1$, *we have*

$$\|\nabla(a - L)^{-1/2}\|_{p,p} \le 2(p^* - 1),$$

where

$$p^* = \max\left\{p, \frac{p}{p-1}\right\}.$$

Suppose that $Ric(L) = Ric + \nabla^2\phi \ge -a$, *where* $a \ge 0$ *is a constant. Then, for all* $p > 1$, *we have*

$$\|\nabla(a - L)^{-1/2}\|_{p,p} \le 2(p^* - 1)^{3/2}.$$

Theorem (Li PTRF2008, Preprint2011)

Let M be a complete Riemannian manifold with $Ric = 0$. Then for all $1 < p < \infty$, we have

$$\|\nabla(-\Delta)^{-1/2}\|_{p,p} \leq 2(p^* - 1).$$

In the case where $Ric \geq 0$, we have

$$\|\nabla(-\Delta)^{-1/2}\|_{p,p} \leq C(p^* - 1)^{3/2}.$$

- Our results extend the estimates of Iwaniec-Martin (Crelles96), Bañuelos-Wang (DMJ95)

$$\|\nabla(-\Delta_{\mathbb{R}^n})^{-/2}\|_{p,p} \leq 2(p^* - 1), \quad \forall p > 1.$$

- At least in the Euclidean and Gaussian cases, the upper bound of type $O(p^* - 1)$ for the L^p norm of the Riesz transform $\nabla(-\Delta)^{-1/2}$ is asymptotically sharp when $p \to 1$ and $p \to \infty$.

Riesz transforms on forms and L^p-Hodge theory

- Riesz transforms associated with the Hodge Laplacian acting on differential forms over complete Riemannian manifolds (Bakry Lect. Notes in Math. 1987).
- Riesz transforms associated with the Laplacian acting on vector bundles over complete Riemannian manifolds (Shigekawa-Yoshida J. Math. Soc. Janpan. 1992).
- Martingale representation formula and L^p-norm estimates of Riesz transforms on differential forms on complete Riemannian manifolds (Li Revista Mat. Iberoamer. 2010).
- Weak L^p-Hodge decomposition and Beurling-Ahlfors transforms on complete Riemannian manifolds (Li Probab. Th. Related Fields 2011)

L^p-estimates and existence theorems of d and $\bar{\partial}$

By the L^p-boundedness of the Riesz transforms and the Riesz potential, we have established

- Strong L^p-Hodge decomposition theorem and the global L^p-estimates and existence theorems of the Cartan-De Rham equation on complete Riemannian manifolds

  $$d\omega = \alpha, \quad d\alpha = 0,$$

  See X.-D. Li J. Funct. Anal. 2009.
- the global $L^{p,q}$-estimates and existence theorems of the Cartan-De Rham equation on complete Riemannian manifolds

  $$d\omega = \alpha, \quad d\alpha = 0,$$

  See X.-D. Li J. Geom. Anal. 2010.
- the global L^p-estimates and existence theorems of the Cauchy-Riemann equation on complete Kähler manifolds

  $$\bar{\partial}\omega = \alpha, \quad \bar{\partial}\alpha = 0.$$

  See X.-D. Li Adv. in Math. 2010.

Witten Laplacian and Bochner-Weitzenböck formula

Let (M, g) be a complete Riemannian manifold, and $\phi \in C^\infty(M)$. Let

$$d\mu = e^{-\phi} dv.$$

Then $\forall \alpha \in C_0^\infty(\Lambda^k T^* M)$, $\beta \in C_0^\infty(\Lambda^{k+1} T^* M)$, we have

$$\begin{aligned} \int_M \langle d\alpha, \beta \rangle e^{-\phi} dv &= \int_M \langle d\alpha, e^{-\phi}\beta \rangle dv \\ &= \int_M \langle \alpha, d^*(e^{-\phi}\beta) \rangle dv \\ &= \int_M \langle \alpha, e^{\phi} d^*(e^{-\phi}\beta) \rangle e^{-\phi} dv, \end{aligned}$$

where d^* denotes the L^2-adjoint of d wrt the volume measure

$$dv(x) = \sqrt{\det g(x)} dx.$$

Let

$$d_\phi^* = e^\phi d e^{-\phi}.$$

Then the following integration by parts (IBP) formula holds

$$\int_M \langle d\alpha, \beta \rangle d\mu = \int_M \langle \alpha, d_\phi^* \beta \rangle d\mu.$$

The Witten Laplacian on M wrt $d\mu = e^{-\phi} dv$ is defined by

$$\Box_\phi = dd_\phi^* + d_\phi^* d.$$

The Bochner-Weitzenböck formula

Let $\Delta = \mathrm{Tr}\nabla^2$ be the covariant Laplacian on $C_0^\infty(\Lambda^\cdot T^*M)$. Let

$$\Delta_\phi = \Delta - \nabla_{\nabla\phi}.$$

Theorem (Bochner-Weitzenböck formula)

Acting on one forms, we have

$$\Box_\phi = -\Delta_\phi + Ric + \nabla^2\phi.$$

In general, acting on k-forms, we have

$$\Box_\phi = -\Delta_\phi + W_k + \Lambda^k\nabla^2\phi,$$

where in a local normal coordinate $(e_1, \ldots, e_n)$ near by x,

$$W_k(x) = \sum_{i<j} e_i^* \wedge i_{e_j} R(e_i, e_j).$$

Weak L^2-Hodge decomposition theorems

Theorem (Kodaira-De Rham)

Let M be a complete Riemannian manifold. Then the Weak L^2-Hodge decomposition theorem holds: for all $k = 0, 1, \ldots, n$,

$$L^2(\Lambda^k T^* M) = \mathrm{Ker}\Box_k \oplus \overline{dC_0^\infty(\Lambda^{k-1} T^* M)} \oplus \overline{d^* C_0^\infty(\Lambda^{k+1} T^* M)},$$

where $\overline{\{\cdot\}}$ denotes the L^2-closure of $\{\cdot\}$ in $L^2(\Lambda^k T^* M)$.

Strong L^2-Hodge decomposition theorems

Theorem (Cheeger, Dodziuk, Donnelly, Gromov, ...)

Let M be a complete Riemannian manifold. Suppose that there exists $\lambda_1 > 0$ such that

$$\|\omega - \mathrm{H}\omega\|_2 \le \lambda_1 \langle\langle \omega, \Box\omega \rangle\rangle, \quad \forall \omega \in C_0^\infty(\Lambda^k T^* M).$$

Then the Strong L^2-Hodge decomposition theorem holds

$$L^2(\Lambda^k T^* M) = \mathrm{Ker}\Box_k \oplus dW^{1,2}(\Lambda^{k-1} T^* M) \oplus d^* W^{1,2}(\Lambda^{k+1} T^* M),$$

where

$$W^{1,2}(\Lambda^k T^* M) = \{\omega \in L^2(\Lambda^k T^* M) : |d\omega| + |d^*\omega| \in L^2(M)\}.$$

Strong L^p-Hodge decomposition theorem

Theorem (Li JFA2009)

Let M be a complete Riemannian manifold, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that

(1) The Riesz transforms $d\Box_k^{-1/2}$ and $d^*\Box_k^{-1/2}$ are bounded in L^p and L^q, i.e., $\exists C_p > 0$ such that: $\forall \omega \in L^r(\Lambda^k T^*M),\ r = p, q$,

$$\|d\Box_k^{-1/2}\omega\|_r + \|d^*\Box_k^{-1/2}\omega\|_r \leq C_p\|\omega\|_r.$$

(2) The Riesz potential $\Box_k^{-1/2}$ is bounded in L^p, i.e., $\exists C_p > 0$ such that

$$\|\Box_k^{-1/2}\omega\|_p \leq C_p\|\omega\|_p, \quad \forall \omega \in L^p(\Lambda^k T^*M), \ \ \Box_k\omega \neq 0.$$

Then the Strong L^p-Hodge decomposition holds: $\forall \omega \in L^p(\Lambda^k T^*M)$,

$$\omega = H\omega + dd^*\Box_k^{-1}\omega + d^*d\Box_k^{-1}\omega.$$

where $H : L^p(\Lambda^k T^*M) \to \mathrm{Ker}\Box_k \cap L^p(\Lambda^k T^*M)$ is the Hodge projection.

L^p-Poincaré inequality on forms

Theorem (Li JFA2009)

Let M be a complete Riemannian manifold, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

Suppose that the Riesz transforms $d\Box_k^{-1/2}$ *and* $d^*\Box_k^{-1/2}$ *are bounded in* L^p *and* L^q*, i.e.,* $\exists C_r > 0$ *such that:* $\forall \omega \in L^r(\Lambda^k T^*M),\ r = p, q,$

$$\|d\Box_k^{-1/2}\omega\|_r + \|d^*\Box_k^{-1/2}\omega\|_r \leq C_p\|\omega\|_r.$$

Then the Riesz potential $\Box_k^{-1/2}$ *is bounded in* L^p*, i.e.,* $\exists C_p > 0$ *such that*

$$\|\Box_k^{-1/2}\omega\|_p \leq C_p\|\omega\|_p, \quad \forall \omega \in L^p(\Lambda^k T^*M),\ \ \Box_k\omega \neq 0,$$

if and only if the L^p*-Poincaré inequality holds on* k*-forms:* $\exists C_p' > 0$ *such that*

$$\|\omega - H\omega\|_p \leq C_p'(\|d\omega\|_p + \|d^*\omega\|_p) \quad \forall \omega \in C_0^\infty(\Lambda^k T^*M),$$

where $H : L^p(\Lambda^k T^*M) \to \mathrm{Ker}\Box_k \cap L^p(\Lambda^k T^*M)$ *is the Hodge projection.*

Riesz transforms on forms

Theorem (Bakry 1987, Li 2010)

Let M be a complete Riemannian manifold. Suppose that

$$W_k + \nabla^2\phi \geq 0, \quad W_{k+1} + \nabla^2\phi \geq 0.$$

Then, $\exists C_k > 0$ *such that:* $\forall p \in (1, \infty)$, $\forall \omega \in L^p(\Lambda^k T^* M, \mu)$,

$$\|d\Box_k^{-1/2}\omega\|_p \leq C_k(p^* - 1)^{3/2}\|\omega\|_p,$$

where $\|\cdot\|_p$ *denotes the* L^p*-norm with respect to* $d\mu = e^{-\phi}dv$.

Semigroup domination and Riesz potential on forms

Theorem (Malliavin, Donnelly-P. Li, Bakry, Elworthy...)

Let M be a complete Riemannian manifold. Suppose that $\exists K \in \mathbb{R}$ such that

$$W_k + \nabla^2\phi \geq K.$$

Then for all $\omega \in C_0^\infty(\Lambda^k T^*M)$,

$$|e^{-t\Box_k}\omega(x)| \leq e^{-Kt}e^{t\Delta}|\omega|(x).$$

Theorem (Bakry-Emery 1986, Li2009)

Let M be a complete Riemannian manifold. Suppose that

$$W_k + \nabla^2\phi \geq \rho > 0.$$

Then $\forall p \in (1, \infty)$, $\forall \omega \in L^p(\Lambda^k T^*M, \mu)$,

$$\|\Box_k^{-1/2}\omega\|_p \leq \frac{1}{\sqrt{\rho}}\|\omega\|_p.$$

where $\|\cdot\|_p$ denotes the L^p-norm with respect to $d\mu = e^{-\phi}dv$.

L^p-estimates and existence theorems of d

Theorem (Li JFA 2009)

Let M *be a complete Riemannian manifold,* $\phi \in C^2(M)$, $d\mu = e^{-\phi} dv$*. Suppose that there exists a constant* $\rho > 0$ *such that*

$$W_k + \nabla^2\phi \geq \rho, \quad W_{k-1} + \nabla^2\phi \geq 0,$$

where W_k *denotes the Weitzenböck curvature operator on* k*-forms. Then, for all* $\alpha \in L^p(\Lambda^k T^*M, \mu)$ *such that*

$$d\alpha = 0,$$

there exists some $\omega \in L^p(\Lambda^{k-1} T^*M, \mu)$ *such that*

$$d\omega = \alpha,$$

and satisfying

$$\|\omega\|_p \leq \frac{C_k (p^* - 1)^{3/2}}{\sqrt{\rho}} \|\alpha\|_p.$$

Beurling-Ahlfors transforms on Riemannian manifolds

Theorem (Li RMI2010)

Let M be a complete Riemannian manifold, $\phi \in C^2(M)$, $d\mu = e^{-\phi} dv$. Suppose that

$$W_k + \nabla^2 \phi \geq 0, \quad W_{k\pm 1} + \nabla^2 \phi \geq 0,$$

where W_k denotes the Weitzenböck curvature operator on k-forms. Then there exists a constant $C_k > 0$ such that for all $p > 1$ and for all $\omega \in L^p(\Lambda^k T^* M, \mu)$, we have

$$\|B_{\mu,k}\omega\|_p \leq C_k (p^* - 1)^3 \|\omega\|_p,$$

where

$$B_{\mu,k}\omega := (dd_\mu^* - d_\mu^* d)\square_{\mu,k}^{-1}\omega$$

is the Beurling-Ahlfors transform with respect to the weighted volume measure μ on k-th forms, and $C_k > 0$ is a constant depending only on k.

Beurling-Ahlfors transforms on Riemannian manifolds

In the case of complete Riemannian manifolds with standard volume measure, we have the following

Theorem (Li PTRF2011)

Let M be a complete Riemannian manifold, $d\nu(x) = \sqrt{\det g(x)}dx$. Suppose that

$$W_k \geq 0,$$

where W_k denotes the Weitzenböck curvature operator on k-forms. Then there exists a constant $C_k > 0$ such that for all $p > 1$ and for all $\omega \in L^p(\Lambda^k T^*M, \nu)$, we have

$$\|B_k\omega\|_p \leq C_k(p^* - 1)^{3/2}\|\omega\|_p,$$

where

$$B_k\omega := (dd^* - d^*d)\Box_k^{-1}\omega$$

is the Beurling-Ahlfors transform with respect to the volume measure ν on k-th forms, and $C_k > 0$ is a constant depending only on k.

Riesz transforms on UMD vector bundles over Riemannian manifolds

Recall that Pisier (1986), Malliavin and Nualart (1994) proved he Riesz transforms associated with the Ornstein-Uhlenbeck operator for the UMD Banach spaces valued functionals is bounded in L^p on the finite and infinite dimensional Gaussian (Wiener) spaces.

This method depends on the rotational invariance of the Gaussian measure. It is an open problem whether this method can be extended to Riesz transforms defined on the complete Riemannian manifolds.

Based on some discussions with Marius Junge and Quanhua Xu, it is very possible that we can extend the Gundy-Varopoulos type martingale representation formulas to the Riesz transform of UMD Banach spaces valued functionals over complete Riemaniann manifolds with suitable curvature condition.

Riesz transforms on UMD Banach spaces valued functionals on manifolds

Theorem (Junge-Li-Xu, work in progress)

Let M be a complete Riemannian manifold, $\phi \in C^2(M)$, and E an UMD Banach space. Suppose that

$$Ric + \nabla^2\phi \geq 0.$$

Then there exists a constant $C(E) > 0$, depending on E but independent of $n = \dim M$ such that for all $1 < p < \infty$,

$$\|\nabla(-L)^{-1/2} f\|_p \leq C(E)(p^* - 1)^{3/2}\|f\|_p,$$

for all $f : M \to E$ with the following form

$$f(x) = \sum_{i=1}^{m} f_i(x) e_i,$$

where $L = \Delta - \nabla\phi \cdot \nabla$, $f_i \in C_0^\infty(M, \mathbb{R})$, $e_1, \ldots, e_m \in E$, $m \in \mathbb{N}$.

Riesz transforms on UMD Banach spaces valued functionals on manifolds

Problem

What happens for the Riesz transforms associated with the Hodge Laplacian on UMD Banach space E-valued differential forms over complete non-compact Riemannian manifolds?

Thank you!