Some Beurling-Fourier algebras are operator algebras

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Weighted convolution algebras

- G: a discrete group.
- $\omega : G \to (0, \infty)$ is called a **weight** if it is sub-multiplicative i.e.

$$\omega(st) \leq \omega(s)\omega(t), \ \ s, t \in G.$$

- $\ell^1(G; \omega)$, a weighted ℓ^1 space equipped with the norm $\|f\|_{\ell^1(G;\omega)} = \sum_{x \in G} \omega(x) \, |f(x)|$, is still **a Banach algebra w.r.t. the convolution** provided that ω is a weight in the above sense. $\ell^1(G; \omega)$ is called a **Beurling algebra on** G.
- (**Example: Polymonial weights**) $G = \mathbb{Z}^d$, $\alpha \geq 0$. $\omega^{\text{poly}}_\alpha(n) = (1 + |n_1| + \cdots + |n_d|)^\alpha$, $n = (n_1, \cdots, n_d) \in \mathbb{Z}^d$.

A result of Varopoulos

- (**Varopoulos, '72**)
  $\ell^1(\mathbb{Z}; \omega_\alpha^{\text{poly}})$ with maximal operator space structure is completely isomorphic to an operator alg. iff $\alpha > \frac{1}{2}$.
- Note that $\ell^1(\mathbb{Z}; \omega_\alpha^{\text{poly}})$ is Aren regular only when $\alpha > 0$.
- (**Ricard, Ghandehari/L/Samei/Spronk, preprint**)
  $\ell^1(\mathbb{Z}^d; \omega_\alpha^{\text{poly}})$ with maximal operator space structure is completely isomorphic to an operator alg. iff $\alpha > \frac{d}{2}$.

Reformulation using co-multilplication

- We begin with the co-multiplication (the adjoint of the convolution map)

$$\Gamma : \ell^{\infty}(G) \to \ell^{\infty}(G \times G)$$

given by $\Gamma(f)(s,t) = f(st)$.

- $(\ell^1(G;\omega))^* = \ell^{\infty}(G;\omega^{-1})$ with the norm

$$\|f\|_{\ell^{\infty}(G;\omega^{-1})} := \left\|\frac{f}{\omega}\right\|_{\infty},$$

so that $\Phi : \ell^{\infty}(G) \to \ell^{\infty}(G;\omega^{-1}),\ \ f \mapsto f\omega$ is an isometry.

Reformulation using co-multilplication: continued

- Using the convolution again on $\ell^1(G;\omega)$ means we will use the same Γ on $\ell^\infty(G;\omega^{-1})$. Then, the isometry Φ gives us the modified co-multiplication

$$\widetilde{\Gamma} : \ell^\infty(G) \to \ell^\infty(G \times G), \;\; f \mapsto \Gamma(f)\Gamma(\omega)(\omega^{-1} \otimes \omega^{-1}).$$

- Note that $\Gamma(\omega)(\omega^{-1} \otimes \omega^{-1}) \leq 1$ iff ω is a weight.
- We would like to do the same procedure in the Fourier algebra setting.

Weighted version of the Fourier algebra $A(G)$

- G : compact group.
- $A(G) = \{f \in C(G) \mid \|f\|_{A(G)} := \sum_{\pi \in \widehat{G}} d_\pi \left\|\widehat{f}(\pi)\right\|_{S^1_{d_\pi}} < \infty\}$, where S^1_n implies the trace class on ℓ^2_n.
- Thus, we have

$$VN(G) \cong \bigoplus_{\pi \in \widehat{G}} M_{d_\pi} \text{ and } A(G) \cong \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi S^1_{d_\pi},$$

so that $A(G)$ is a one of the simplest non-commutative L^1-spaces.
- The representation picture of G suggests us a **simple model for a weight**.
- $A(G;\omega) := \{f \in C(G) \mid$
$$\|f\|_{A(G;\omega)} := \sum_{\pi \in \widehat{G}} d_\pi \omega(\pi) \left\|\widehat{f}(\pi)\right\|_{S^1_{d_\pi}} < \infty\}.$$

Weighted version of the Fourier algebra $A(G)$: continued

- The co-multiplication this time is given by
$$\Gamma : VN(G) \to VN(G \times G), \ \ \lambda(x) \mapsto \lambda(x) \otimes \lambda(x),$$
where $\lambda(x)$ is the left translation operator acting on $L^2(G)$.
- For $\omega : \widehat{G} \to (0, \infty)$ we associate an operator
$$W = (W(\pi)), \ \ W(\pi) = \omega(\pi) id_{M_{d_\pi}}.$$
- We consider the following weighted spaces
$VN(G; W^{-1}) := \{AW : A \in VN(G)\}$ with the norm
$\|AW\|_{VN(G;W^{-1})} = \|A\|_{VN(G)}$ and
$A(G; W) := \{W^{-1}\phi : \phi \in A(G)\}$ with the norm
$\left\|W^{-1}\phi\right\|_{A(G;W)} = \|\phi\|_{A(G)}$.
- Clearly $A(G; W) \cong A(G; \omega)$.
- $\Phi : VN(G) \to VN(G; W^{-1}), \ \ A \mapsto AW$ is an (complete) isometry.

Weighted version of the Fourier algebra $A(G)$: continued 2

- If we use the same Γ on $VN(G; W^{-1})$, then by applying Φ we get a modified co-multiplication

$$\widetilde{\Gamma} : VN(G) \to VN(G \times G), \;\; A \mapsto \Gamma(A)\Gamma(W)(W^{-1} \otimes W^{-1}).$$

- We say that $\omega : \widehat{G} \to (0, \infty)$ is a **weight** if

$$\Gamma(W)(W^{-1} \otimes W^{-1}) \leq I.$$

- Then $A(G; W)$ is a (completely contractive) Banach algebra under the pointwise multiplication. We call $A(G; W)$ a **Beurling-Fourier algebra on** G.

Examples of weights

- We need to transfer Γ to the setting on $\bigoplus_{\pi\in\widehat{G}} M_{d_\pi}$. For any $A = (A(\pi))_{\pi\in\widehat{G}}$ we have

$$\Gamma(A)(\pi,\pi') \cong \bigoplus_{\sigma\subset\pi\otimes\pi'} A(\sigma), \quad \pi,\pi'\in\widehat{G},$$

  where $\sigma\subset\pi\otimes\pi'$ implies that $\sigma\in\widehat{G}$ appears in the decomposition of $\pi\otimes\pi'$.
- Thus, $\omega:\widehat{G}\to(0,\infty)$ is a **weight** if and only if

$$\omega(\sigma)\leq\omega(\pi)\omega(\pi')$$

  for every $\sigma\subset\pi\otimes\pi'$.
- $\omega_\alpha(\pi) = d_\pi^\alpha$, $\pi\in\widehat{G}$, **the dimension weight of order** α.
- G: connected Lie group, S: a finite generating set in $\widehat{G}$. $\tau_S(\pi) =$ (length function) the least number k with $\pi\in S^{\otimes k}$. $\omega_S^\alpha(\pi) = (1+\tau_S(\pi))^\alpha$, **the polynomial weight of order** α.

Some Beurling-Fourier algebras are operator algebras

- (**Blecher, '95**)
  A c.c. Banach alg. $\mathcal{A}$ is completely isomorphic to an operator alg. iff the multiplication map m extends to a completely bounded map $m : \mathcal{A} \otimes_h \mathcal{A} \to \mathcal{A}$.
- $A(G, \omega)$ with its natural operator space structure is completely isomorphic to an operator alg. iff the modified co-multiplication $\tilde{\Gamma}$ extends to a completely bounded map

$$\tilde{\Gamma} : VN(G) \to VN(G) \otimes_{eh} VN(G),$$

where $VN(G) \otimes_{eh} VN(G) \cong (A(G) \otimes_h A(G))^*$.

Positive directions

- Since $\tilde{\Gamma} : VN(G) \to VN(G) \bar{\otimes} VN(G)$ is a complete contraction and $\widetilde{\Gamma}(A) = \Gamma(A)\Gamma(W)(W^{-1} \otimes W^{-1})$ we can get positive results when $\Gamma(W)(W^{-1} \otimes W^{-1})$ is a **"multiplier"** from $VN(G)\bar{\otimes} VN(G)$ into $VN(G) \otimes_{eh} VN(G)$.
- (**Non-commutative Littlewood multiplier: Ghandehari/L/Samei/Spronk, preprint**)
Elements in $VN(G)\bar{\otimes} L^2_r(VN(G))$ and $L^2_c(VN(G))\bar{\otimes} VN(G)$ are left and right cb-multipliers from $VN(G)\bar{\otimes} VN(G)$ into $VN(G) \otimes_{eh} VN(G)$, where H_r and H_c are row and column Hilbert spaces for a Hilbert space H.
- We hope to find the decomposition

$$\Gamma(W)(W^{-1} \otimes W^{-1}) = T_1 + T_2,$$

$$T_1 \in L^2_c(VN(G))\bar{\otimes} VN(G) \text{ and } T_2 \in VN(G)\bar{\otimes} L^2_r(VN(G)).$$

Positive directions: continued

- Let $T = \Gamma(W)(W^{-1} \otimes W^{-1})$, then

$$T(\pi, \pi') \cong \bigoplus_{\sigma \subset \pi \otimes \pi'} \frac{\omega(\sigma)}{\omega(\pi)\omega(\pi')} id_{M_{d_\sigma}}$$

- When G is a compact connected Lie group and $\omega = \omega_\alpha$ we have

$$\frac{\omega(\sigma)}{\omega(\pi)\omega(\pi')} \lesssim \frac{1}{(1+\tau_S(\pi))^\alpha} + \frac{1}{(1+\tau_S(\pi'))^\alpha},$$

so that $T \lesssim T_1 + T_2$ with
$T_1 = \left(\bigoplus_{\pi \in \widehat{G}} \frac{1}{(1+\tau_S(\pi))^\alpha} id_{M_{d_\pi}}\right) \otimes 1_{VN(G)}$ and
$T_2 = 1_{VN(G)} \otimes \left(\bigoplus_{\pi' \in \widehat{G}} \frac{1}{(1+\tau_S(\pi'))^\alpha} id_{M_{d_{\pi'}}}\right)$

Positive directions: continued 2

- $\left\| \tilde{T}_2 \right\|_{VN(G)\bar{\otimes}L^2_r(VN(G))} \lesssim \left(\sum_{\pi \in \widehat{G}} \frac{d_\pi^2}{(1+\tau_S(\pi))^{2\alpha}} \right)^{\frac{1}{2}} < \infty$
  if $\alpha > \frac{d(G)}{2}$ (a well-known Lie theory).
- (**Ghandehari/L/Samei/Spronk, preprint**)
  G: connected Lie group, S: a canonical generating set
  $A(G, \omega_S^\alpha)$ is completely isomorphic to an operator algebra if $\alpha > \frac{d(G)}{2}$.
- Recall that when $G = \mathbb{T}^d$ the above condition is actually an if and only if condition.

Positive directions: the case of dimension weights

- (**Ghandehari/L/Samei/Spronk, preprint**)
  $G = SU(n)$, $n \leq 5$
  $A(G, \omega^\alpha)$ is completely isomorphic to an operator algebra if $\alpha > \frac{d(G)}{2}$.
- The above result is related to the following conjecture on representations of $SU(n)$.
- (**Conjecture**, true for $n \leq 5$)
  Let $\pi_\lambda, \pi_\mu, \pi_\nu \in \widehat{SU(n)}$ with $\pi_\nu \subset \pi_\lambda \otimes \pi_\mu$ Then we have

$$\frac{\dim \pi_\nu}{\dim \pi_\lambda \dim \pi_\mu} \lesssim C_n \left(\frac{1}{\lambda_1 + 1} + \frac{1}{\mu_1 + 1} \right).$$

- However, $A(G, \omega^\alpha)$ is never completely isomorphic to an operator algebra for $G = U(n)$ (non-simple).

Negative directions

- (**Restriction of weights to subgroups**)
  H : a closed subgroup of G, $\omega : \widehat{G} \to (0,\infty)$: a weight.
  We get a weight $\omega_H : \widehat{H} \to (0,\infty)$ defined by
  $$\omega_H(\rho) = \inf\{\omega(\pi) \mid \rho \subset \pi|_H\}.$$
  Then $A(H;\omega_H)$ is a (completely contractive) Banach algebra quotient of $A(G;\omega)$.
- (**Ghandehari/L/Samei/Spronk, preprint**)
  $G = SU(n)$, $H \cong T^{n-1}$ the maximal torus
  $(\omega_\alpha)_H \cong \omega^{\text{poly}}_{(n-1)\alpha}$ and $(\omega^\alpha_S)_H \cong \omega^{\text{poly}}_\alpha$.
- $A(SU(n),\omega^\alpha_S)$ is not completely isomorphic to an operator alg. if $\alpha \leq \frac{n-1}{2}$.
- $A(SU(n),\omega_\alpha)$ is not completely isomorphic to an operator alg. if $\alpha \leq \frac{1}{2}$.

Some consequences

- $A(G;\omega_{2^k})$ is known to be a unital closed subalgebra of $A(G^{(2k)})$, where $G^{(2k)} = G \times \cdots \times G$, $2k$-times.
- $A(SU(n);\omega_{2^k})$ is a unital closed subalgebra of $A(SU(n)^{(2k)})$ which are isomorphic to an operator algebra for big enough k.
- Hopefully, we get interesting non-selfadjoint operator algebras associated to groups.

Further directions for Beurling-Fourier algebras

- Is $\alpha > \frac{\dim G}{2}$ optimal?
- Non-central weights by extension procedure.
- The case of compact quantum groups.
- Non-compact groups.