Square function estimates and dilation properties of operators on noncommutative L^p-spaces

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Square functions on classical L^p-spaces

Let (Ω, μ) be a measure space and let $1 < p < \infty$.
Let $T\colon L^p(\Omega) \to L^p(\Omega)$.

For any $x \in L^p(\Omega)$, we consider

$$\|x\|_T = \Big\| \Big(\sum_{k=1}^{\infty} k \big| T^k(x) - T^{k-1}(x) \big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p}.$$

This may be finite or infinite.

- Such square functions appear in Stein's work (from 60's).

He showed that if :

(i) $T\colon L^1(\Omega) \longrightarrow L^1(\Omega)$ is a contraction ;

(ii) $T\colon L^\infty(\Omega) \longrightarrow L^\infty(\Omega)$ is a contraction ;

(iii) $T\colon L^2(\Omega) \longrightarrow L^2(\Omega)$ is positive in the Hilbertian sense ;

(iv) $T(1) = 1$,

then for all $1 < p < \infty$, the operator

$$T\colon L^p(\Omega) \longrightarrow L^p(\Omega)$$

satisfies a **square function estimate**

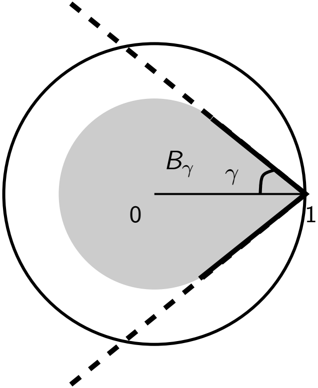
$$\|x\|_T \leqslant C_p \|x\|_{L^p}.$$

- These operators are the so-called positive **Markov operators**.
- These square function estimates are related to :
  - square functions associated to martingales ;
  - maximal inequalities.

Reference : E.M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Ann. Math. Studies, Princeton, University Press, 1970.

Stolz domains

For any angle $0 < \gamma < \frac{\pi}{2}$, we let B_γ be the interior of the convex hull of 1 and of the disc of radius $\sin\gamma$ centered at 0.



For any function $\varphi\colon B_\gamma \to \mathbb{C}$, we set

$$\|\varphi\|_{\infty,B_\gamma} = \sup\{|\varphi(z)| : z \in B_\gamma\}.$$

Main 'commutative' result

Let $\mathcal{P}$ be the algebra of complex polynomials.

Theorem (2010)

For any bounded operator $T\colon L^p(\Omega) \to L^p(\Omega)$, the following assertions are equivalent.

(i) There exists a constant $C > 0$ such that

$$\|x\|_T \leqslant C\|x\|_p \qquad \text{and} \qquad \|y\|_{T^*} \leqslant C\|y\|_{p'}$$

for any $x \in L^p(\Omega)$ and any $y \in L^{p'}(\Omega)$.

(ii) There exist an angle $\gamma < \frac{\pi}{2}$ and a constant $K > 0$ such that

$$\forall\, \varphi \in \mathcal{P}, \qquad \|\varphi(T)\colon L^p(\Omega) \longrightarrow L^p(\Omega)\| \leqslant K\|\varphi\|_{\infty,B_\gamma}.$$

Ritt operators

Let X be a Banach space. We say that $T\colon X \to X$ is a **Ritt operator** if there exist two constants $C_0, C_1 > 0$ such that

$$\forall\, k \geqslant 0, \qquad \|T^k\| \leqslant C_0;$$

$$\forall\, k \geqslant 1, \qquad k\big\|T^k - T^{k-1}\big\| \leqslant C_1.$$

This is equivalent to the 'Ritt condition' :

$$\sigma(T) \subset \overline{\mathbb{D}}$$

and there exists $C > 0$ such that

$$\forall\, \lambda \notin \overline{\mathbb{D}}, \qquad \|(\lambda - T)^{-1}\| \leqslant \frac{C}{|\lambda - 1|}\,.$$

This implies that

$$\exists\, \gamma \in \big(0, \frac{\pi}{2}\big) \;\Big|\; \qquad \sigma(T) \subset \overline{B_\gamma}.$$

- For any $\gamma \in \left(0, \frac{\pi}{2}\right)$, there exists a constant $K > 0$ such that

$$k|z^k - z^{k-1}| \leqslant K$$

for any $z \in B_\gamma$ and for all $k \geqslant 1$.

- Therefore the condition

(ii) There exist an angle $\gamma < \frac{\pi}{2}$ and a constant $K > 0$ such that

$$\forall \varphi \in \mathcal{P}, \qquad \|\varphi(T) \colon L^p(\Omega) \longrightarrow L^p(\Omega)\| \leqslant K\|\varphi\|_{\infty, B_\gamma}$$

implies that :

T is a Ritt operator.

A special class (joint work with Q. Xu)

Theorem (2010)

Let $1 < p < \infty$ and let $T\colon L^p(\Omega) \to L^p(\Omega)$ be a **positive contraction**. If T is a Ritt operator, then there exists a constant $C > 0$ such that

$$\forall\, x \in L^p(\Omega), \qquad \Big\| \Big(\sum_{k=1}^{\infty} k \big| T^k(x) - T^{k-1}(x) \big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \leqslant C \|x\|_{L^p}.$$

- This extends Stein's Theorem (on positive Markov operators).

- Since $T^*\colon L^{p'}(\Omega) \to L^{p'}(\Omega)$ is also a positive contraction, this implies a B_γ-functional calculus.

Application to maximal inequalities

For any $1 < p < \infty$ and any $T\colon L^p(\Omega) \to L^p(\Omega)$, set

$$M_n(T) = \frac{1}{n+1} \sum_{k=0}^{n} T^k\,.$$

Theorem (Akcoglu, '75)

If T is a positive contraction, then

$$\forall\, x \in L^p(\Omega), \qquad \Big\| \sup_{n\geqslant 0} \big| M_n(T)x \big| \Big\|_p \leqslant C_p \|x\|_p.$$

Corollary

Let $T\colon L^p(\Omega) \to L^p(\Omega)$ be a positive contraction. If T is a Ritt operator, then there is a constant $K > 0$ such that

$$\forall\, x \in L^p(\Omega), \qquad \Big\| \sup_{n\geqslant 0} \big| T^n(x) \big| \Big\|_p \leqslant K \|x\|_p.$$

Proof (just to understand the role of square functions)

- This goes back to Stein's work on Markov operators.
- By an Abel tranformation,

$$\sum_{k=1}^{n} k(T^k - T^{k-1}) = nT^n - \sum_{k=0}^{n-1} T^k \,.$$

Hence for any $x \in L^p(\Omega)$,

$$\begin{aligned}
T^n(x) &= M_{n-1}(T)x + \frac{1}{n}\sum_{k=1}^{n} k\big(T^k(x) - T^{k-1}(x)\big).\\
|T^n(x)| &\leqslant \big|M_{n-1}(T)x\big| + \frac{1}{n}\sum_{k=1}^{n} k\big|T^k(x) - T^{k-1}(x)\big|\\
&\leqslant \big|M_{n-1}(T)x\big| + \frac{1}{n}\Big(\sum_{k=1}^{n} k\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{n} k\big|T^k(x) - T^{k-1}(x)\big|^2\Big)^{\frac{1}{2}}\\
&\leqslant \big|M_{n-1}(T)x\big| + \Big(\sum_{k=1}^{\infty} k\big|T^k(x) - T^{k-1}(x)\big|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Noncommutative L^p-spaces

Let (M, τ) be a von Neumann algebra equipped with a normal semifinite faithful trace.

For any $1 \leqslant p < \infty$, let $L^p(M)$ denote the associated noncommutative L^p-space.

Any element x of $L^p(M)$ is a (possibly unbounded operator) and we set

$$|x| = (x^*x)^{\frac{1}{2}}.$$

For any $x \in M \cap L^p(M)$,

$$\|x\|_{L^p(M)} = \big(\tau(|x|^p)\big)^{\frac{1}{p}}$$

Square functions on noncommutative L^p-spaces

Consider $T\colon L^p(M) \to L^p(M)$ and let $x \in L^p(M)$.

If $2 \leqslant p < \infty$ we set

$$\|x\|_T = \max\Big\{\Big\|\Big(\sum_{k=1}^{\infty} k\big|T^k(x) - T^{k-1}(x)\big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p},\qquad \Big\|\Big(\sum_{k=1}^{\infty} k\big|\big(T^k(x) - T^{k-1}(x)\big)^*\big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}\Big\}.$$

If $1 < p \leqslant 2$ we set

$$\|x\|_T = \inf\Big\{\Big\|\Big(\sum_k |u_k|^2\Big)^{\frac{1}{2}}\Big\|_p + \Big\|\Big(\sum_k |v_k^*|^2\Big)^{\frac{1}{2}}\Big\|_p\Big\},$$

where the infimum runs over all possible decompositions

$$u_k + v_k = k^{\frac{1}{2}}\big(T^k(x) - T^{k-1}(x)\big) \qquad (k \geqslant 1).$$

Main noncommutative result

Theorem

Let $T\colon L^p(M) \to L^p(M)$ be a bounded operator, with $1 < p < \infty$. The following are equivalent.

(i) There exist an angle $\gamma < \frac{\pi}{2}$ and a constant $K > 0$ such that

$$\forall\, \varphi \in \mathcal{P}, \qquad \|\varphi(T)\colon L^p(M) \longrightarrow L^p(M)\| \leqslant K\|\varphi\|_{\infty,B_\gamma}.$$

(ii) T is an R-Ritt operator and there exists a constant $C > 0$ such that

$$\|x\|_T \leqslant C\|x\|_{L^p} \qquad \text{and} \qquad \|y\|_{T^*} \leqslant C\|y\|_{L^{p'}}$$

for any $x \in L^p(M)$ and any $y \in L^{p'}(M)$.

Question

Which Ritt operators $L^p(M) \to L^p(M)$ do satisfy the above conditions ?

- Positive or completely positive contractions ?

- Case when
  $T\colon M \to M$ and $T\colon L^1(M) \to L^1(M)$ are (complete) contractions
  and
  $T\colon L^2(M) \to L^2(M)$ is positive in the Hilbertian sense ?

Schur multipliers (d'après Junge-LeM-Xu)

Let $T\colon B(\ell^2) \to B(\ell^2)$ be a *contractive* Schur multiplier, given by

$$[c_{ij}]_{i,j\geqslant 1} \mapsto [t_{ij}c_{ij}]_{i,j\geqslant 1}$$

- For any $1 \leqslant p < \infty$, let S^p be the p-Schatten class.
- Then T extends to a contraction $T\colon S^p \to S^p$.

Theorem

Assume that t_{ij} is a <u>nonnegative real number</u> for any $i, j \geqslant 1$.

For any $1 < p < \infty$, there exists an angle $\gamma < \frac{\pi}{2}$ and a constant $K > 0$ such that

$$\forall\, \varphi \in \mathcal{P}, \qquad \|\varphi(T)\colon S^p \longrightarrow S^p\| \leqslant K\|\varphi\|_{\infty,B_\gamma}.$$

Consequently, we have estimates :

$$\|x\|_T \leqslant C\|x\|_{S^p}, \qquad x \in S^p.$$

Noncommutative positive Markov operators (d'après Junge-Ricard-Shlyakhtenko)

We consider (M, τ) with $\tau(1) = 1$.
In this case, $M \subset L^p(M)$ for any $1 \leqslant p < \infty$.

A linear map $T\colon M \to M$ is called a **Markov operator** if

(i) T is completely positive.
(ii) T is unital, i.e. $T(1) = 1$.
(iii) T is trace preserving, i.e. $\tau \circ T = \tau$.

Then for any $1 \leqslant p < \infty$, such an operator (uniquely) extends to a contraction

$$T_p\colon L^p(M) \longrightarrow L^p(M).$$

We say that T is *selfadjoint* (resp. *positive*) if moreover T_2 is selfadjoint (resp. positive in the Hilbertian sense).
If T is a selfadjoint Markov operator, then T^2 is a positive Markov operator.

Noncommutative positive Markov operators (continued)

Theorem

Let $T: M \to M$ be a positive Markov operator.

For any $1 < p < \infty$, there exists an angle $\gamma < \frac{\pi}{2}$ and a constant $K > 0$ such that

$$\forall \varphi \in \mathcal{P}, \qquad \|\varphi(T_p): L^p(M) \longrightarrow L^p(M)\| \leqslant K\|\varphi\|_{\infty, B_\gamma}.$$

Consequently, we have estimates :

$$\|x\|_{T_p} \leqslant C\|x\|_{L^p}, \qquad x \in L^p(M).$$

This is based on :

• A recent work of Junge-Ricard-Shlyakhtenko showing that for T as above, then

$$e^{-t(I-T_p)}: L^p(M) \longrightarrow L^p(M)$$

is *factorizable* for any $t \geqslant 0$;

Noncommutative loose dilations

Let $1 < p < \infty$ and let $T\colon L^p(M) \to L^p(M)$.
We say that T admits a **loose dilation** if there is another von Neumann algebra N, a surjective isometry $U\colon L^p(N) \to L^p(N)$ and two bounded operators

$$J\colon L^p(M) \to L^p(N) \qquad \text{and} \qquad Q\colon L^p(N) \to L^p(M)$$

such that

$$T^n = QU^nJ$$

for any $n \geqslant 0$.

A square function characterization (joint work with C. Arhancet)

Theorem

Let $1 < p < \infty$ and let $T\colon L^p(M) \to L^p(M)$ be an R-Ritt operator. The following assertions are equivalent.

(i) T admits a noncommutative loose dilation.

(ii) There exists a constant $C > 0$ such that

$$\|x\|_T \leqslant C\|x\|_{L^p} \qquad \text{and} \qquad \|y\|_{T^*} \leqslant C\|y\|_{L^{p'}}$$

for any $x \in L^p(M)$ and $y \in L^{p'}(M)$.

An alternative square function ($p < 2$)

For any $1 < p < \infty$, for any $T\colon L^p(M) \to L^p(M)$ and for any $x \in L^p(M)$, set :

$$\|x\|_{T,c} = \Big\| \Big(\sum_{k=1}^{\infty} k \big| T^k(x) - T^{k-1}(x) \big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p}$$

and

$$\|x\|_{T,r} = \Big\| \Big(\sum_{k=1}^{\infty} k \big| \big(T^k(x) - T^{k-1}(x) \big)^* \big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p}.$$

Assume that $1 < p < 2$ and set

$$\|x\|_{T,r+c} = \inf\big\{ \|x_1\|_{T,c} + \|x_2\|_{T,r} \ : \ x = x_1 + x_2 \big\}.$$

Then we formally have

$$\|x\|_T \leqslant \|x\|_{T,r+c}.$$

An alternative square function (continued)

Indeed if $x = x_1 + x_2$, then for any $k \geqslant 1$,

$$k^{\frac{1}{2}}\big(T^k(x) - T^{k-1}(x)\big) = u_k + v_k$$

with

$$u_k = k^{\frac{1}{2}}\big(T^k(x_1) - T^{k-1}(x_1)\big) \quad \text{and} \quad v_k = k^{\frac{1}{2}}\big(T^k(x_2) - T^{k-1}(x_2)\big)$$

- What about a converse estimate ?
- When do we have an estimate

$$\|x\|_{T,r+c} \leqslant C\|x\|_{L^p} \, ?$$

Positive results (by C. Arhancet)

Theorem

Let $T\colon L^p(M) \to L^p(M)$ be a Ritt operator, with $1 < p < 2$. Assume that there exists an angle $\gamma < \frac{\pi}{2}$ and a constant $K > 0$ such that

$$\forall\, \varphi \in \mathcal{P}, \qquad \|\varphi(T)\colon L^p(M) \longrightarrow L^p(M)\|_{cb} \leqslant K\|\varphi\|_{\infty,B_\gamma}.$$

Then there exists a constant $C > 0$ such that

$$\|x\|_{T,r+c} \leqslant C\|x\|_{L^p}, \qquad x \in L^p(M).$$

That is, any $x \in L^p(M)$ has a decomposition $x = x_1 + x_2$ in $L^p(M)$ with

$$\Big\|\Big(\sum_{k=1}^{\infty} k\big|T^k(x_1) - T^{k-1}(x_1)\big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \leqslant C\|x\|_{L^p}$$

and $$\Big\|\Big(\sum_{k=1}^{\infty} k\big|\big(T^k(x_2) - T^{k-1}(x_2)\big)^*\big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \leqslant C\|x\|_{L^p}.$$

Conclusion

The above 'completely bounded functional calculus property' and hence the resulting square function estimate hold for :

- Contractive positive Schur multipliers ;
- Positive Markov operators.