THE INVERTIBILITY OF TOEPLITZ OPERATORS WITH NONCOMMUTING SYMBOLS

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Outline of talk.

- Noncommutative $H^p(M)$-spaces.
- Toeplitz operators with noncommuting symbols.
- Devinatz's classical result.
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- A noncommutative Helson-Szegö theorem.
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1. Introduction

In the early 1960's the following simple setting describing a large class of function algebras exhibiting H^∞-like behaviour, was isolated:

Let X be a probability space, and let A be a weak closed unital-subalgebra of $L^\infty(X)$, such that:*

$$\int fg = \int f \int g, \qquad f, g \in A. \tag{1}$$

(Here the given measure on X is the *representing measure* of the multiplicative functional $f \to \int f$ on A.)

Write $[\mathcal{S}]_p$ for the closure of a set $\mathcal{S} \subset L^p$ in the p-norm. For an algebra of the above type we formally define $H^p = [A]_p$. Letting $A_0 = \{f \in A : \int f = 0\}$, we similarly define $H_0^p = [A_0]_p$.

Theorem 1.1. *For such A, t.f.a.e.:*

(i) $A + \bar{A}$ *is weak* dense in* $L^\infty(X)$.

(ii) A *is 'logmodular': if* $b \in L^\infty(X)$ *with* $b >> 0$ *then* b *is a uniform limit of terms of the form* $|a|^2$ *for an invertible* $a \in A$.

(iii) *Szegö's formula:* $\forall g \in L^1_+(X)$, $\exp \int \log g = \inf\{\int |1-f|^2 g : f \in A, \int f = 0\}$.

(iv) *Beurling-Nevanlinna factorization: Every* $f \in L^2(X)$ *such that* $f \notin [fA_0]_2$ *has an 'inner-outer factorization'* $f = uh$, *with* u *unimodular and* $h \in [A]_2$ *such that* $1 \in [hA]_2$.

(v) *Riesz-Szegö theorem: For any* $0 < p \leq \infty$ *and* $f \in L^p$, *we have* $\int \log |f| dm > -\infty$ *iff* $f = uh$ *for some unimodular function* $u \in L^\infty$ *and outer element* $h \in H^p$. *(Here* outer *means the closure in* L^p *of* hA *is all of* H^p.*)*

The algebras described by the above theorem are the so-called *weak*-Dirichlet algebras*. The theory of these algebras goes on to show that the weak* Dirichlet algebras mirror many properties of classical H^∞.

The noncommutative framework:

$\mathbf{L^p}, \mathbf{L^\infty}$: We assume that M is a von Neumann algebra possessing a faithful normal tracial state τ. (So τ is a weak*-continuous positive linear functional for which $x = 0$ whenever $\tau(x^*x) = 0$, and $\tau(xy) = \tau(yx)$ for all $x, y \in M$.) Such algebras will be refered to as finite von Neumann algebras.

For such von Neumann algebras we may define the noncommutative $L^p(M)$ spaces as the completion of M under the (p)-norm $\| \cdot \| = \tau(| \cdot |^p)^{1/p}$.

Conditional Expectation: For any von Neumann subalgebra N of such an M, there also exists a weak*-continuous contractive projection $\Psi : M \to N$ satisfying $\tau \circ \Psi = \tau$. (The so-called faithful normal conditional expectation onto N with respect to τ.)

Defining H^∞: A *tracial subalgebra* of M is a weak* closed unital subalgebra A of M such that the trace preserving faithful normal conditional expectation $\Phi : M \to A \cap A^* = \mathcal{D}$ satisfies:

$$\Phi(a_1 a_2) = \Phi(a_1)\,\Phi(a_2), \quad a_1, a_2 \in A. \tag{2}$$

If $\mathcal{D} = A \cap A^*$ is one dimensional, we call A *antisymmetric*.

A tracial subalgebra is said to be a *finite maximal subdiagonal algebra* of M, if $A + A^*$ weak* dense in M. (These are our noncommutative H^∞'s.)

For these algebras essentially the same set of equivalences as in the classical case (with some fascinating diversification of structure here and there) pertains.

Theorem 1.2. *For a tracial subalgebra A of M, the following conditions are equivalent:*

(i) *A is maximal subdiagonal: $\overline{A+A^*}^{w*} = M$.*

(ii) *A is logmodular: if $b \in M_+$ is invertible then b is a uniform limit of terms of the form a^*a for invertible $a \in A$.*

(iii) *A satisfies a Szegö-like formula formulated in terms of the Fuglede-Kadison determinant.*

(iv) *Beurling-Nevanlinna-like factorization property.*

(v) *A Riesz-Szegö theorem for elements of $L^2(M)$.*

Emergent meta-theorem: *The classical theory of H^p-spaces is far more algebraic in nature than anticipated.*

Set $A_0 = A \cap \mathrm{Ker}(\Phi)$. For maximal subdiagonal algebras the analogue of H^p is $[A]_p$, the closure of A in the noncommutative L^p space $L^p(M)$, for $p \geq 1$. We write $H^p_0(M)$ for $[A_0]_p$.

As in the classical case, whenever $1 < p < \infty$ we here too have that the spaces $H^p(M)$ and $L^p(\mathcal{D})$ satisfy the complementation property $L^p(M) = H^p_0(M) \oplus L^p(\mathcal{D}) \oplus H^p_0(M)^*$ where $H^p(M) = H^p_0(M) \oplus L^p(\mathcal{D})$. For the case $p = 2$ this is easy to see. The fact that $\tau \circ \Phi = \tau$ ensures A_0, $\mathcal{D}$ and A_0^* are mutually orthogonal with respect to the inner product $\langle a, b \rangle = \tau(a^* b)$. It is then a matter of using the weak* density of $A + A^*$ to see that $H^2_0(M) \oplus L^2(\mathcal{D}) \oplus H^2_0(M)^*$ is all of $L^2(M)$.

Some examples of such algebras include

(1) All the classical weak*-Dirichlet algebras;
(2) Upper triangular matrices in $M_n(\mathbb{C})$ with Φ the projection onto the diagonal matrices, and $\tau = \frac{1}{n}\mathrm{Tr}$;
(3) (Arveson) Let G be a countable discrete group with a linear ordering invariant under left multiplication. The subalgebra generated by G_+ in the group von Neumann algebra of G is a subdiagonal algebra.

2. Toeplitz operators with noncommuting symbols

We briefly review some of the contributions of Marsalli and West [IEOT, 1998] on this topic.

Given $a \in M$ we may define the left multiplication operator

$$L_a : L^2(M) \to L^2(M) : b \mapsto ab.$$

Now let P denote the orthogonal projection on $L^2(M)$ mapping onto $H^2(M)$ and along $H^2_0(M)^*$. If we restrict L_a to $H^2(M)$ and then compose the result with P, we get the so-called Toeplitz operator on $H^2(M)$ with symbol a, namely

$$T_a : H^2(M) \to H^2(M) : b \mapsto P(ab).$$

It is a simple matter to see that

$$\|T_a\| \leq \|L_a\| = \|a\|_\infty.$$

Given $a, b \in M$ it is a simple matter to see that $L_a L_b = L_{ab}$. Without some additional assumptions, the same formula is however not always valid for Toeplitz operators. The following result holds:

Proposition 2.1. *Let $a, b \in M$ be given. If either $b \in A = H^\infty(M)$ or $a \in A^* = H^\infty(M)^*$, we get that $T_a T_b = T_{ab}$.*

The following important result may also be found in the paper of Marsalli and West:

Theorem 2.2 (Noncommutative Hartman-Winter spectral inclusion)**.** *For any $a \in M$ we have that $\sigma(a) = \sigma(L_a) \subset \sigma(T_a)$.*

Challenge: With a respectable theory of noncommutative H^p spaces in place and an established concept of Toeplitz operators for this framework, the challenge we now face is the following question. *Can we as in the classical setting give a structural characterisation of those symbols a for which T_a is an invertible operator?*

3. Devinatz's classical result

In the context of the classical $H^2(\mathbb{T})$ space, Allen Devinatz established the following elegant characterisation of invertible Toeplitz operators around 1964.

Theorem 3.1. *Given $f \in L^\infty(\mathbb{T})$, the Toeplitz operator T_f is invertible if and only the following conditions hold:*

- $\mathrm{ess\,inf}|f| > 0$;
- *there exists a $g \in H^\infty \cap (H^\infty)^{-1}$ and some $\epsilon > 0$ so that $|Arg(gf)| \leq \frac{\pi}{2} - \epsilon$.*

His strategy was to first reduce the problem to one of describing those unitaries u in L^∞ for which the T_u is invertible, by means of the following result:

Theorem 3.2. *Given $f \in L^\infty(\mathbb{T})$, let $f = u|f|$ be the polar decomposition of f. The Toeplitz operator T_f is invertible if and only the following conditions hold:*

- $\mathrm{ess\,inf}|f| > 0$;
- *T_u is invertible;*
- *u is of the form $\overline{h}/h$ for some h with $h, 1/h \in H^2$.*

Given some $w \in L^1(\mathbb{T})^+$ we may write $L^2(w)$ for the Hilbert space generated by means of the (semi-) inner product $\langle f, g\rangle_w = \int_{\mathbb{T}} \overline{g} f w\, dm$. (As usual we of course reduce to equivalence classes in the case where $\|\cdot\|_w$ turns out to be a seminorm.)

The closures of $H^\infty(\mathbb{T})$ and $\overline{H_0^\infty(\mathbb{T})}$ in the $\|\cdot\|_w$ norm will be denoted by $H^2(w)$ and $\overline{H_0^2(w)}$. The angle between these closed subspaces of $L^2(w)$ is defined to be

$$\arccos \rho$$

where ρ is given by

$$\rho = \sup\{|\langle f, g\rangle_w| : f \in A_0, g \in A^*, \|f\|_w \leq 1, \|g\|_w \leq 1\}.$$

For these spaces to be at a positive angle, we need to have that $0 \leq \rho < 1$. It is not too difficult to show that requiring $\rho < 1$ is equivalent to requiring the existence of some $K > 1$ for which

$$\|f\|_w^2 + \|g\|_w^2 \leq K\|f+g\|_w^2$$

(Simply set $K = (1-\rho)^{-1}$.). Hence positivity of the angle ensures that $H^2(w) + \overline{H_0^2(w)}$ is a Banach space direct sum of $H^2(w)$ and $\overline{H_0^2(w)}$.

To any function $f \in \Re(H^\infty)$ we may associate a uniquely determined function $\widetilde{f}$ for which $f + i\widetilde{f} \in H^\infty$ and $\int_{\mathbb{T}} \widetilde{f}\, dm = 0$. The map $f \to \widetilde{f}$ is called the *(harmonic) conjugation map*. It is not difficult to extend this to a map on $\Re(H^\infty) + i\Re(H^\infty)$.

Theorem 3.3. *Given a finite Borel measure μ on $\mathbb{T}$, the spaces $H^2(\mu)$ and $\overline{H_0^2(\mu)}$ are at a positive angle (with respect to μ) if and only if the conjugation map $f \to \widetilde{f}$ (f a trigonometric polynomial) is bounded in $L^2(\mathbb{T}, \mu)$ norm.*

Using these ideas the required description of the unitaries for which T_u is invertible, was achieved by Devinatz in the following result:

Theorem 3.4. *Let $u \in L^\infty(\mathbb{T})$ be a unitary of the form $u = \overline{h}/h$ for some h with $h, 1/h \in H^2$, and write $w = |h|^2$. Then T_u is invertible if and only if the spaces $H^2(w)$ and $\overline{H_0^2(w)}$ are at a positive angle.*

Although thus far we have much evidence to indicate the importance of measures for which $H^2(\mu)$ and $\overline{H_0^2(\mu)}$ are at a positive angle, we still have no structural information regarding these measures. For this we need the very remarkable Helson-Szegö theorem:

Theorem 3.5 (Helson-Szegö)**.** *Let μ be a finite positive measure on $\mathbb{T}$ with Lebesgue decomposition $d\mu = w dm + d\mu_s$. Then $H^2(\mu)$ and $\overline{H_0^2(\mu)}$ are at a positive angle if and only if $\mu_s = 0$ and $w \in L^1(\mathbb{T})$ is of the form $\log(w) = a + \widetilde{b}$ for some $a, b \in L^\infty(\mathbb{T})$ with $\|b\|_\infty < \frac{\pi}{2}$.*

4. Pousson's contribution

As in the setting of Devinatz, L^∞ and H^2 will here denote the classical spaces $L^\infty(\mathbb{T})$ and $H^2(\mathbb{T})$.

The context: Given some $F = [f_{ij}] \in M_n(L^\infty)$ we may define an associated Toeplitz operator on the column space $M_{n,1}(H^2) = C_n(H^2)$, by setting $T_F([a_k]) = P([f_{ij}][a_k])$ where P is the orthogonal projection of $C_n(L^2)$ onto $C_n(H^2)$.

Similar to Devinatz, Pousson reduced the characterisation of invertibility to the case of unitary elements of $M_n(L^\infty)$ by means of the following result:

Theorem 4.1. *Given $F = [f_{ij}] \in M_n(L^\infty)$, T_F is invertible if and only if F admits of a factorisation $F = UK$ where $U \in M_n(L^\infty)$ is unitary, $K \in M_n(H^\infty)$ outer, and both T_U and T_K invertible. In this case we have that* $\operatorname{ess\,inf} |\det(F)| > 0$.

The characterisation of the unitaries $U \in M_n(L^\infty)$ for which T_U is invertible, follows from a combination of the following two facts:

Theorem 4.2. *Let $U \in M_n(L^\infty)$ be a unitary.*

(1) *If T_U is invertible, it is of the form $U = G_0^* G_1$ where G_0, G_0^{-1}, G_1 and G_1^{-1} all belong to $M_n(H^2)$ and $G_1 G_1^* = (G_0 G_0^*)^{-1}$.*

(2) *For any unitary $U \in M_n(L^\infty)$ of the above form, T_U is invertible if and only if the subspaces $M_n(H^2)(W)$ and $M_n(\overline{H_0^2})(W)$ are at a positive angle, where $W = G_0 G_0^*$. (Here $M_n(H^2)(W)$ the closure of $M_n(H^\infty)$ in $M_n(L^2)(W)$ where $M_n(L^2)(W)$ is constructed using an inner product suitably weighted with W.)*

Remark 4.3 (Deconstructing Helson-Szegö). *Suppose we are given some $w \in L^1(\mathbb{T})^+$ for which the spaces $H^2(w)$ and $\overline{H_0^2(w)}$ are at a positive angle.*

- *The first step is to show that then $\int \log(w)\, dm > -\infty$, which enables us to apply the classical Riesz-Szegö theorem to obtain a factorisation of the form $w = uh$ for some unimodular function $u \in L^\infty$ and outer element $h \in H^1$.*
- *Using this factorisation the outerness of h and a judicious use of duality, enables one to show that quantity $\rho = \sup\{|\langle f, g\rangle_w| : f \in A_0, g \in A^*, \|f\|_w \leq 1, \|g\|_w \leq 1\}$ may be identified with $\rho = \inf_{g\in H^\infty} \|u - g\|_\infty$.*
- *To the above fact one now applies the following elegant lemma:*
  For a real-valued measurable function ψ, $\inf_{g\in H^\infty} \|e^{-i\psi} - g\|_\infty < 1$ if and only if there exists an $\epsilon > 0$ and a $k_0 \in H^\infty$ so that $|k_0| \geq \epsilon$ a.e. with $|\psi + arg(k_0)| \leq \frac{\pi}{2} - \epsilon$. *The operators a and b in the formulation of the Helson-Szegö theorem then correspond to $a = -\log|k_0|$ and $b = -\psi - arg(k_0)$*

Proposition 4.4 (Mildly noncommutative Helson-Szegö)**.** *Given* $W \in M_n(L_1)^+$ *for which* $\log(det(W))$ *is summable, there exists outer elements* $A, B \in M_n(H^2)$ *with* $W = AA^* = B^*B$*. The subspaces* $M_n(H^2)(W)$ *and* $M_n(\overline{H_0^2})(W)$ *are then at a positive angle if and only if for the unitary* $U = B^{-1}A^* = B^*A^{-1}$ *we can find a* $G \in M_n(H^\infty)$ *and an* $\alpha > 0$ *such that*

$$\alpha + G^*G \leq U^*G + G^*U.$$

How about the passage to subdiagonal algebras? Can the above approach carry over? In passing to the von Neumann algebra context we do not only have "stronger" noncommutativity, but also incur a further significant complication which is that instead of the very nice situation of having $H^\infty(\mathbb{D}) \cap \overline{H^\infty(\mathbb{D})} = \mathbb{C}\mathbb{1}$, the intersection $H^\infty(M) \cap H^\infty(M)^* = \mathcal{D}$ can be highly non-trivial!!

5. Prelude to the noncommutative result (outers and peak sets)

Definition 5.1 (Noncommutative geometric mean)**.** *On any finite von Neumann algebra with faithful normal tracial state τ, the* Fuglede-Kadison determinant *is defined by setting*

$$\Delta(a) = \exp \tau(\log(|a|))$$

if $a \in M$ is invertible, with

$$\Delta(a) = \inf_{\epsilon>0} \exp \tau(\log(|a| + \epsilon 1))$$

otherwise. (The definition extends to include all the $L^p(M)$'s ($p > 0$). Brown 1986; Haagerup-Schultz 2006)

Many important classical formulae can be expressed in this context using this determinant. For example the Jensen inequality (valid for any $p > 0$) reads that $\Delta(a) \geq \Delta(\Phi(a))$ for all $a \in H^p(M)$.

Theorem 5.2. (Noncommutative Verblunsky/Kolmogorov-Krein) *Let ω be a positive linear functional on M, and let $\omega_n = \tau(h\,\cdot)$ ($h \in L^1(M)_+$) and ω_s be its normal and singular parts respectively. Then*

$$\Delta(h) = \inf\{\omega(|a|^2) : a \in A, \Delta(\Phi(a)) \geq 1\}.$$

The infimum remains unchanged if we also require a to be invertible in A.

(BL initially proved this for the case $\dim(\mathcal{D}) < \infty$. Later Bekjan and Xu managed to remove this restriction.)

Recall that

- an element h of $H^p(M)$ $(p \geq 1)$ is said to be an *outer* function if $[hA]_p = H^p(M)$, and strongly outer if in addition $\Delta(h) > 0$.

We assume $p \geq 1$ throughout the rest of this section. By using the generalised Jensen inequality, the Szegö formula and the observation that $h \in H^p(M)$ is outer iff $1 \in [hA]_p$, we can now prove the following:

Theorem 5.3 (Characterisation of outers)**.** *Let A be a subdiagonal algebra, let $1 \leq p \leq \infty$ and $h \in H^p$. If h is outer then $\Delta(h) = \Delta(\Phi(h))$. If $\Delta(h) > 0$, this condition is also sufficient for h to be outer.*

With some delicate analysis, a Helson-Lowdenslager type trick, and a sharpening of the known Riesz factorization results for subdiagonal algebras, one can now prove a noncommutative version of the famous Riesz-Szegö theorem.

Theorem 5.4 (Noncommutative Riesz-Szegö)**.** *If A is a maximal subdiagonal algebra, and $f \in L^p(M)$ then $\Delta(f) > 0$ iff $f = uh$ for a unitary u and an outer $h \in H^p$ with positive determinant. Moreover, this factorization is unique up to a unitary in $\mathcal{D}$.*

The classical peak-set theorem for the disc algebra runs as follows:

Theorem 5.5. *Let $E \subset \mathbb{T}$ be a compact subset. Then $m(E) = 0$ if and only if E is a peak set for $A_0(\mathbb{D})$ (i.e. there exists $f \in A_0(\mathbb{D})$ with $f = 1$ on E, and $|f(z)| < 1$ on $\overline{\mathbb{D}} \backslash E$.)*

A very useful consequence of the peak set theorem states that if μ is a finite complex Borel measure on $\mathbb{T}$ which is singular with respect to m, then there exists analytic polynomials p_n such that

- $|p_n(z)| \leq 1$ whenever $|z| \leq 1$,
- $p_n\mu \to \|\mu\|$,
- $p_n \to 0$ m-ae.

A faithful noncommutative version of the peak-set for subdiagonal algebras, is too much to hope for. However recently (ArXiV: 2008) Ueda proved a very respectable noncommutative version of the previously mentioned consequence:

Theorem 5.6. *Let A be a finite maximal subdiagonal subalgebra of M, and ω a nonzero singular functional on M. Then we can find a contractive element $a \in A$, and a projection $p \in M^{\star\star}$ so that*

- a^n *converges to p in the $\sigma(M^{\star\star}, M^\star)$-topology as $n \to \infty$;*
- $p(|\omega|) = |\omega|(\mathbb{1})$*;*
- a^n *converges to 0 in the $\sigma(M, M^\star)$-topology as $n \to \infty$.*

6. A noncommutative Helson-Szegö theorem

Given a state ω, we define the angle between the spaces A^* and A_0 to be $\arccos \rho$ where

$$\rho = \sup\{|\omega(b^*a)| : a \in A_0, b \in A^*, \omega(|a|^2) \leq 1, \omega(|b|^2) \leq 1\}.$$

In general $0 \leq \rho \leq 1$.

We can express this in Hilbert space language by looking at the Hilbert space $\mathfrak{h}_\omega = L^2(\omega)$ constructed in the GNS construction for ω. The subspaces A^* and A_0 embed canonically into $L^2(\omega)$ by means of the operation $a \to \pi_\omega(a)\Omega_\omega$. The angle between A^* and A_0 as defined above, is then the same as the angle between the closed subspaces $\overline{\pi_\omega(A^*)\Omega_\omega}$ and $\overline{\pi_\omega(A_0)\Omega_\omega}$ of $L^2(\omega)$. Specifically

$$\rho = \sup\{|\langle \pi_\omega(a)\Omega_\omega, \pi_\omega(b)\Omega_\omega \rangle| : a \in A_0, b \in A^*, \|\pi_\omega(a)\Omega_\omega\| \leq 1, \|\pi_\omega(b)\Omega_\omega\| \leq 1\}.$$

Proposition 6.1. *Let $\mathcal{D} = A \cap A^*$ be finite dimensional, and let ω be a state for which $\rho < 1$. Then ω is of the form $\omega = \tau(g\cdot)$ for some $g \in L^1_+(M)$.*

Proof. It is not too difficult to show that $\rho = 1$ if $\overline{\pi_\omega(A_0)\Omega_\omega} \cap \overline{\pi_\omega(A^*)\Omega_\omega} \neq \{0\}$.

Hence suppose that $\omega_s \neq 0$ and that

$$\overline{\pi_\omega(A_0)\Omega_\omega} \cap \overline{\pi_\omega(A^*)\Omega_\omega} = \{0\}.$$

Select an orthogonal projection p in $M^{\star\star}$ and $a \in A$ with $\|a\| \leq 1$ so that

- a^n converges to p in the weak*-topology on $M^{\star\star}$;
- $\omega_s(p) = \omega_s(\mathbb{1})$ (here ω_s is identified with its canonical extension to $M^{\star\star}$);
- a^n converges to 0 in the weak*-topology on M.

Since the expectation Φ is weak*-continuous on M, $\Phi(a^n)$ is weak* convergent to 0. But then the finite dimensionality of $\mathcal{D}$ ensures that $\Phi(a^n)$ converges to 0 in norm.

By the Riesz representation theorem there exists some $x_p \in \mathfrak{h}_\omega$ so that

$$p(\langle \cdot, \eta\rangle) = \langle x_p, \eta\rangle \quad \text{for every} \quad \eta \in \mathfrak{h}_\omega.$$

There exists a central projection p_0 in $\pi_\omega(M)''$ for which $(p_0\pi_\omega, p_0\mathfrak{h}_\omega, p_0\Omega_\omega)$ and $((\mathbb{1} - p_0)\pi_\omega, (\mathbb{1} - p_0)\mathfrak{h}_\omega, (\mathbb{1} - p_0)\Omega_\omega)$ are respectively copies of the GNS representations of ω_n and ω_s. The bullets above, then translate to the statements that

- $\pi_\omega(a^n)\Omega_\omega$ converges to x_p in the weak-topology on $\mathfrak{h}_\omega$;
- $\langle x_p, (\mathbb{1} - p_0)\Omega_\omega\rangle = \omega_s(\mathbb{1})$.
- $x_p \in \overline{\pi_\omega(A_0)\Omega_\omega}$

By verifying and using the fact that $(a^*)^n$ also converges to p, we may now conclude that

$$x_p \in \overline{\pi_\omega(A^*)\Omega_\omega} \cap \overline{\pi_\omega(A_0)\Omega_\omega} = \{0\}.$$

But if $\omega_s \neq 0$, this would contradict the second bullet! Therefore $\omega_s = 0$. □

The support projection of any self-adjoint element f of $\widetilde{M}$ will be denoted by $s(f)$.

Lemma 6.2. *For any $g \in L^1_+(M)$ we have that*

$$s(\Phi(g)) \geq s(g).$$

Lemma 6.3. *Let e be a non-zero projection in $\mathcal{D}$. Then eAe is a finite maximal subdiagonal subalgebra of eMe (equipped with the trace $\tau_e(\cdot) = \frac{1}{\tau(e)}\tau(\cdot)$) with diagonal $eAe \cap (eAe)^* = e\mathcal{D}e$.*

Definition 6.4. *Adopting the notation of the previous two lemmas, given a non-zero element $g \in L^1_+(M)$, we define $\Delta_\Phi(g)$ to be the determinant of $s_\Phi g s_\Phi$ regarded as an element of $(s_\Phi M s_\Phi, \tau_{s(\Phi(g))})$*

Proposition 6.5. *Let $\mathcal{D} = A \cap A^*$ be finite dimensional, and let $g \in L^1_+(M)$ be a norm-one element for which the state $\omega = \tau(g\cdot)$ satisfies $\rho < 1$. Then $\Delta_\Phi(g) > 0$.*

Proof. Assuming that $s(\Phi(g)) = \mathbb{1}$, we suppose by way of contradiction that $\Delta(g) = 0$. By the Szegö formula for subdiagonal algebras we then have that

$$0 = \Delta(g) = \inf\{\tau(g|a - d|^2) : a \in A_0, d \in \mathcal{D}, \Delta(d) \geq 1\}.$$

Thus there exist sequences $\{a_n\} \subset A_0$ and $\{d_n\} \subset \mathcal{D}$ with $\Delta(d_n) \geq 1$ for all n, so that

$$\tau(g|a_n - d_n|^2) \to 0 \quad \text{as} \quad n \to \infty.$$

Now reduce to the case where the d_n's are positive and invertible.

Next consider $\widetilde{d_n} = \frac{1}{\|d_n\|} d_n$ and $\widetilde{a_n} = \frac{1}{\|d_n\|} a_n$ and conclude from the above that

$$\pi_g(\widetilde{d_n}) \to \pi_g(\widetilde{d_0}) \neq 0$$

and

$$\|\pi_g(\widetilde{a_n}) - \pi_g(\widetilde{d_0})\| \to 0.$$

This ensures that $\overline{\pi_g(A_0)} \cap \overline{\pi_g(A^*)}$, and hence that $\rho = 1$. $\square$

Significance: Knowing that ω must be of the form $\tau(g\cdot)$ with $\Delta_\Phi(g) > 0$ whenever $\rho < 1$, gives us access to the all-important Riesz-Szegö theorem.

Still to do: Characterise those normal states $\omega = \tau(g\cdot)$ for which $\Delta_\Phi(g) > 0$ and $\rho < 1$.

Theorem 6.6. *Let $g \in L^1_+(M)$ be given with $\|g\|_1 = 1$, and denote $s(\Phi(g))$ by s_Φ. For the state $\omega = \tau(g\cdot)$, the two conditions $\rho < 1$ and $\Delta_\Phi(g) > 0$ hold if and only if the following two conditions hold:*

- *there exists a unitary u in $s_\Phi M s_\Phi$, and outers $h_L, h_R \in H^2(s_\Phi M s_\Phi)$, such that $g = h_R u h_L = |h_L|^2 = |h_R^*|^2$;*
- *for some $k_0 \in s_\Phi A s_\Phi$ and some $0 < \alpha \leq 1$ we have $\alpha s_\Phi + |k_0|^2 \leq u^* k_0 + k_0^* u$.*

(As demonstrated earlier, if $\mathcal{D}$ is finite dimensional, then in the above equivalence we may dispense with the restrictions that ω is normal, and that $\Delta_\Phi(g) > 0$.)

Proof. Without loss of generality assume that $s_\Phi(g) = 1\!\!1$. We prove only the "only if" part. Let g satisfy the conditions $\rho < 1$ and $\Delta(g) > 0$. Since $\Delta(g^{1/2}) = \Delta(g)^{1/2} > 0$, the noncommutative Riesz-Szegö theorem ensures that there exists outer elements $h_L, h_R \in H^2(M)$ and unitaries $v_L, v_R \in M$ for which

$$g^{1/2} = v_L^* h_L = h_R v_R^* \qquad g^{1/2} = |h_L| = |h_R^*|.$$

For any $a \in A_0$, $b \in A^*$ we have $\langle \pi_g(a)\Omega_g, \pi(b)\Omega_g \rangle = \tau(gb^*a) = \tau(v_R^* v_L^* h_L b^* a h_R)$. So

$$\begin{aligned}
\rho &= \sup\{|\tau(gb^*a)| : a \in A_0, b \in A^*, \tau(g|a|^2) \leq 1, \tau(g|b|^2) \leq 1\} \\
&= \sup\{|\tau(v_R^* v_L^* h_L b^* a h_R)| : a \in A_0, b \in A^*, \tau(|ah_R|^2) \leq 1, \tau(|h_L b^*|^2) \leq 1\} \\
&= \sup\{|\tau(v_R^* v_L^* f_1 f_2)| : f_1 \in H^2(M), f_2 \in H_0^2(M), \|f_1\|_2 \leq 1, \|f_2\|_2 \leq 1\} \\
&= \sup\{|\tau(v_R^* v_L^* F)| : F \in H_0^1(M), \|F\|_1 \leq 1\}.
\end{aligned}$$

(In the above computation f_1 and f_2 are approximated by $h_L b^*$ and ah_R respectively.)

By duality we now have that

$$\rho = \inf\{\|v_R^* v_L^* - k\|_\infty : k \in s_\Phi A s_\Phi\}.$$

Hence $\rho < 1$ if and only if there exists $k_0 \in A$ so that

$$1 > \|v_R^* v_L^* - k_0\|_\infty = \|\mathbb{1} - v_R k_0 v_L\|_\infty.$$

This last property can now be translated to the the required conclusion. □

A different translation of the property verified above yields the following variant:

Theorem 6.7. *Let $g \in L^1_+(M)$ be given with $\|g\|_1 = 1$, and denote $s(\Phi(g))$ by s_Φ. Consider the state $\omega = \tau(g\cdot)$. If $\Delta_\Phi(g) > 0$, then for $\omega = \tau(g\cdot)$ we have*

$$\rho < 1 \quad \Leftrightarrow \quad g = h_R b_g h_L + a_g$$

where

$$a_g \in H_1(M); \quad b_g \in M; \quad h_R, h_L \in L^2(M) \quad \text{with} \quad \|b_g\|_\infty < 1; \quad |h_L|^2, |h_R^*|^2 \leq g.$$

We may choose h_R, h_L to be strongly outer with $g = |h_L|^2 = |h_R^*|^2$.

7. Invertibility of Toeplitz operators

Theorem 7.1. *Let $a \in M$ be given. A necessary and sufficient condition for T_a to be invertible is that it can be written in the form $a = uk$ where $k \in A^{-1}$, and $u \in M$ is a unitary for which T_u is invertible.*

Suppose that $a \in M$ is indeed of the form $a = uk$ where $k \in A^{-1}$, and $u \in M$ is a unitary. It is a simple exercise to see that then T_k is invertible with inverse $T_{k^{-1}}$. Since we then have that $T_a T_{k^{-1}} = T_u$ and $T_u T_k = T_a$, it is clear that T_a will then be invertible if and only if T_u is invertible. Thus to fully characterise invertibility of Toeplitz operators, we still need to characterise invertibility of Toeplitz operators with unitary symbols.

Proof. We outline the proof of necessity.

- T_a to be invertible means there exists $g \in H^2(M)$ so that $T_a g = \mathbb{1}$. Equivalently there exists some $h \in H^2_0(M)$ so that $ag = \mathbb{1} + h^*$.
- Now use the generalised Jensen inequality to see that $\Delta(ag) \geq \Delta(\mathbb{1}) = 1$ and hence that $\Delta(|a|^{1/2}) > 0$.
- Now use the noncommutative Riesz-Szegö theorem to find an outer element $f \in H^2$ with $|f|^2 = |a|$.
- The tricky part of the proof is to now show that T_f is actually invertible. From this and the Hartmann-Winter spectral inclusion one can then deduce that $|a|$ is strictly positive. Applying Arveson's factorisation theorem then proves the result.

□

Lemma 7.2. *Let $u \in M$ be a unitary. A necessary condition for T_u to be invertible is that it is of the form $u = (g_1^*)^{-1} d g_0^{-1}$ where g_0, g_1 are strongly outer elements of $H^2(M)$ and d a strongly outer element of $L^2(\mathcal{D})$ related by the conditions that*

$$d = \Phi(g_0) = \Phi(g_1^*), \quad dg_0^{-1}, d^* g_1^{-1} \in H^2(M) \quad \textit{and} \quad g_0^* g_0 = d^*(g_1^* g_1)^{-1} d.$$

Proof. We again outline the proof.

- Using the invertibility of T_u and T_{u^*}, we select $g_0, g_1 \in H^2(M)$ so that $T_u g_0 = \mathbb{1} = T_{u^*} g_1$, or equivalently $ug_0 = \mathbb{1} + h_0^*$ and $u^* g_1 = \mathbb{1} + h_1^*$.
- Now use the generalised Jensen inequality to conclude from these equalities that $\Delta(g_0), \Delta(g_1) > 0$. This ensures that g_0 and g_1 are each unitarily equivalent to an outer element. The challenge is to show that they are actually outer themselves. This proves to be a very non-trivial exercise which uses the full scope of the characterisation of outers achieved thus far.
- The final challenge is to show that $g_1^* u g_0$ is actually uniformly bounded and belongs to $\mathcal{D}$. Setting $d = g_1^* u g_0$ then proves the result.

□

Lemma 7.3. *Let $u \in M$ be unitary of the form described in the previous lemma. Then T_u is invertible if and only if A^* and A_0 are at a positive angle to each other with respect to the functional $\tau(w\cdot)$, where $w = g_0^* g_0 = d^*(g_1^* g_1)^{-1} d$.*

If we further combine the above lemma with the noncommutative Helson-Szegö theorem, we end up with the required structural characterisation of the symbols of invertible Toeplitz operators.

Theorem 7.4. *Let $u \in M$ be unitary of the form described in the previous lemma. Then T_u is invertible if and only if there exists a $k \in A$ and some $0 < \alpha \leq 1$ with $\alpha + |k|^2 \leq u^* k + k^* u$.*