On the optimality of entanglement witnesses

2012.06.08., Wuhan

Seung-Hyeok Kye
Seoul National University

1. What is entanglement ?
2. Positive linear maps and entanglement witnesses
3. Optimality: Decomposable cases
4. Optimality: Indecomposable cases

- S.-H. Kye, Necessary conditions for optimality of decomposable entanglement witness, Rep. Math. Phys., to appear. arXiv:1108.0456
- K.-C. Ha and S.-H. Kye, Entanglement witnesses arising from exposed positive linear maps, Open Syst. Inf. Dyn. 18 (2011), 323-337. arXiv:1108.0130
- K.-C. Ha and S.-H. Kye, Entanglement witnesses arising from Choi type positive linear maps, preprint. arXiv:1205.2921

1. What is entanglement ?

A positive semi-definite matrix in $M_m \otimes M_n = M_m(M_n)$ is said to be *separable* if it is the sum of rank one projectors onto product vectors in $\mathbb{C}^m \otimes \mathbb{C}^n$. A *product vector* is a simple tensor $\xi \otimes \eta \in \mathbb{C}^m \otimes \mathbb{C}^n$.

Therefore, if A is separable then it is of the form

$$A = \sum_i z_i z_i^* = \sum_i |z_i\rangle\langle z_i| \in M_m \otimes M_n$$

with product vectors $z_i = \xi_i \otimes \eta_i \in \mathbb{C}^m \otimes \mathbb{C}^n$.
We denote by $\mathbb{V}_1$ the convex cone consisting of all separable ones.

A positive semi-definite matrix in $M_m \otimes M_n$ is said to be *entangled* if it is not separable.

By the relation

$$(\xi \otimes \eta)(\xi \otimes \eta)^* = \xi\xi^* \otimes \eta\eta^*$$

we see that

$$\mathbb{V}_1 = M_n^+ \otimes M_m^+,$$

and so, entanglement consists of

$$(M_n \otimes M_m)^+ \setminus M_n^+ \otimes M_m^+.$$

Note that $(A \otimes B)^+ = A^+ \otimes B^+$ for *commutative* C^*-algebras A and B.

$z = e_1 \otimes e_1 + e_2 \otimes e_1 = (1,0,1,0)^{\mathrm{t}} = (e_1 + e_2) \otimes e_1 = |00\rangle + |10\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is a product vector, and so

$$zz^* = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is separable. But,
$z = e_1 \otimes e_1 + e_2 \otimes e_2 = (1,0,0,1)^{\mathrm{t}} = |00\rangle + |11\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is not a product vector, and so

$$zz^* = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

is entangled.

If rank > 1 then it is very difficult in general to determine if a given positive semi-definite matrix in $M_m \otimes M_n$ is separable or entangled.

For $A \in M_m \otimes M_n$, define the *partial transpose* $A^\tau \in M_m \otimes M_n$ by

$$(X \otimes Y)^\tau = X^{\mathrm{t}} \otimes Y,$$

for $X \in M_n$ and $Y \in M_m$. Then

$$\left(\sum_{ij=1}^{m} e_{ij} \otimes x_{ij} \right)^\tau = \sum_{ij=1}^{m} e_{ji} \otimes x_{ij} = \sum_{ij=1}^{m} e_{ij} \otimes x_{ji}$$

So, partial transpose is nothing but the block-wise transpose.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^\tau = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}^\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For $\xi \in \mathbb{C}^m$ and $\eta \in \mathbb{C}^n$, we have

$$\begin{aligned}[(\xi \otimes \eta)(\xi \otimes \eta)^*]^\tau &= [\xi\xi^* \otimes \eta\eta^*]^\tau \\ &= (\xi\xi^*)^{\mathrm{t}} \otimes \eta\eta^* \\ &= \bar{\xi}\bar{\xi}^* \otimes \eta\eta^* \\ &= (\bar{\xi} \otimes \eta)(\bar{\xi} \otimes \eta)^*\end{aligned}$$

The partial transpose of a rank one projection onto a *product vector* is again a rank one projection, especially positive semi-definite.

Therefore, if $A \in M_n \otimes M_m$ is separable then its partial transpose A^τ is also positive semi-definite.

The product vector $\bar{\xi} \otimes \eta \in \mathbb{C}^m \otimes \mathbb{C}^n$ is called the *partial conjugate* of $\xi \otimes \eta$.

This gives us a simple necessary condition, called the PPT (positive partial transpose) criterion for separability, as was observed by Choi (1982) and Peres (1996). Denote by

$$\mathbb{T} = \{A \in (M_n \otimes M_m)^+ : A^\tau \in (M_n \otimes M_m)^+\}.$$

With this notation, the PPT criterion says that

$$\mathbb{V}_1 \subseteq \mathbb{T}.$$

The equality holds if and only if $(m, n) = (2, 2), (2, 3)$ or $(3, 2)$, by Woronowicz (1976) and Choi (1982).

2. Positive linear maps and entanglement witnesses

A linear map $\phi : M_m \to M_n$ is *positive* if it send positive (positive semi-definite) matrices into positive matrices, where M_n is the C^*-algebra of all $n \times n$ matrices.
Elementary examples are

$$\phi : X \to \sum V_i^* X V_i + \sum W_i^* X^{\mathrm{t}} W_i$$

where V_i and W_i are $m \times n$ matrices. Those positive maps are said to be *decomposable*.

Question: Is every positive map decomposable ?
No, by M.-D. Choi (1975) for $m = n = 3$.

S. L. Woronowicz (1976) showed that the following are equivalent:

(i) Every positive map from M_2 into M_n is decomposable.

(ii) If Q is of PPT then there exists a product vector $x \otimes y \in \mathbb{C}^2 \otimes \mathbb{C}^n$ in the range of Q such that its partial conjugates $\bar{x} \otimes y$ belongs to the range of Q^τ, that is, $\mathbb{T} \subset \mathbb{V}_1$.

- (i) is true for $n = 2$ by Størmer (1963)
- Woronowicz showed that (ii) is also true for $n = 3$, and gave a counterexample for $n = 4$. This is an example of $2 \otimes 4$ PPT entangled edge state of type $(5, 5)$, in the current terminology.

M.-D. Choi (1980) gave an example of PPT states which is not in the cone $M_3^+ \otimes M_3^+$. This is the $3 \otimes 3$ PPT entangled edge state of type $(4, 4)$.

E. Størmer (1982) showed that the following are equivalent:

- A linear map ϕ from a C^*-algebra $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ is decomposable.
- If $[x_{ij}]$ and $[x_{ji}]$ belong to $M_n(\mathcal{A})^+$ then $[\phi(x_{ij})]$ is positive in $M_n(\mathcal{B}(\mathcal{H}))$, for every $n = 1, 2, \ldots$

He gave an example of $[x_{ij}] \in M_3(M_3)^+$ with $[x_{ji}] \in M_3(M_3)^+$, in order to give a very short proof of the indecomposability of the Choi's example. This the $3 \otimes 3$ PPT entangled edge state of type $(6, 7)$.

For a bounded linear map ϕ from a C^*-algebra A into $\mathcal{B}(\mathcal{H})$, $x \in A$ and $y \in \mathcal{T}(\mathcal{H})$, define

$$\langle x \otimes y, \phi \rangle = \mathrm{Tr}\,(\phi(x) y^{\mathrm{t}}),$$

where Tr denotes the usual trace. This gives rise to a bilinear pairing between two spaces $\mathcal{B}(A, \mathcal{B}(\mathcal{H}))$ of all bounded linear operators from a C^*-algebra A into $\mathcal{B}(\mathcal{H})$ and the projective tensor product $A \hat{\otimes} \mathcal{T}(\mathcal{H})$.

This pairing was used by Woronowicz (1976) for the above-mentioned result, Størmer (1986) to study extendibility of positive linear maps. The predual cones of $\mathbb{P}_s[A, \mathcal{B}(\mathcal{H})]$ and $\mathbb{P}^s[A, \mathcal{B}(\mathcal{H})]$ with respect to the above pairing have been determined by T. Itoh (1986).

We restrict ourselves to the cases of matrix algebras, to get the duality between the space $M_m \otimes M_n$ and the space $\mathcal{L}(M_m, M_n)$. For $A = \sum_{i,j=1}^m e_{ij} \otimes a_{ij} \in M_m \otimes M_n$ and a linear map $\phi \in \mathcal{L}(M_m, M_n)$, we have

$$\langle A, \phi \rangle = \sum_{i,j=1}^{m} \mathrm{Tr}\,(\phi(e_{ij})\, a_{ij}^{\mathrm{t}}) = \sum_{i,j=1}^{m} \langle a_{ij}, \phi(e_{ij}) \rangle,$$

where the bilinear form in the right-side is given by $\langle X, Y \rangle = \mathrm{Tr}\,(YX^{\mathrm{t}})$ for $X, Y \in M_n$. Therefore, this pairing is nothing but

$$\langle A, \phi \rangle = \mathrm{Tr}\,(AC_\phi^{\mathrm{t}}) = \mathrm{Tr}\,(C_\phi A^{\mathrm{t}})$$

for two matrices A and C_ϕ in $M_m \otimes M_n$ with the usual trace, where

$$C_\phi = (\mathrm{id}_m \otimes \phi) \left(\sum_{i,j=1}^{m} e_{ij} \otimes e_{ij} \right) = \sum_{i,j=1}^{m} e_{ij} \otimes \phi(e_{ij}).$$

The correspondence $\phi \mapsto C_\phi$ is called the Jamiołkowski-Choi isomorphism.

Identify the vector space $\mathbb{C}^m \otimes \mathbb{C}^n$ with the space $M_{m\times n}$ of all $m \times n$ matrices. Every vector $z \in \mathbb{C}^m \otimes \mathbb{C}^n$ is uniquely expressed by

$$z = \sum_{i=1}^{m} e_i \otimes z_i \in \mathbb{C}^m \otimes \mathbb{C}^n, \qquad z_i = \sum_{k=1}^{n} z_{ik} e_k \in \mathbb{C}^n, \qquad i = 1, 2, \dots, m.$$

Then we get $z = [z_{ik}] \in M_{m\times n}$. This identification

$$\sum_{i=1}^{m} e_i \otimes \left(\sum_{k=1}^{n} z_{ik} e_k \right) \longleftrightarrow [z_{ik}]$$

is an inner product isomorphism from $\mathbb{C}^m \otimes \mathbb{C}^n$ onto $M_{m\times n}$.

$$\xi \otimes \bar{\eta} \leftrightarrow \xi\eta^* \in M_{m\times n},$$
$$e_i \otimes e_j \leftrightarrow e_{ij} \in M_{m\times n},$$
$$\sum_i e_i \otimes e_i \leftrightarrow \mathrm{Identity}.$$

For $s = 1, 2, \ldots, m \wedge n$, we define the convex cones $\mathbb{V}_s$ and $\mathbb{V}^s$ in $M_m \otimes M_n$ by

$$\mathbb{V}_s(M_m \otimes M_n) = \mathrm{conv}\,\{zz^* \in M_m \otimes M_n : \mathrm{rank}\, z \le s\},$$
$$\mathbb{V}^s(M_m \otimes M_n) = \mathrm{conv}\,\{(zz^*)^\tau \in M_m \otimes M_n : \mathrm{rank}\, z \le s\}.$$

Since rank one matrix $z \in M_{m\times n}$ corresponds to a product vector $z \in \mathbb{C}^m \otimes \mathbb{C}^n$, the cone $\mathbb{V}_1$ coincides with the convex cone generated by all separable states.

Eom+K (2000) showed

$$\phi \in \mathbb{P}_s \iff \langle A, \phi\rangle \ge 0 \text{ for each } A \in \mathbb{V}_s,$$
$$A \in \mathbb{V}_s \iff \langle A, \phi\rangle \ge 0 \text{ for each } \phi \in \mathbb{P}_s,$$

and similarly for the pair $(\mathbb{V}^t, \mathbb{P}^t)$, where $\mathbb{P}_s$ (resp. $\mathbb{P}^t$) denotes the cone of all s-positive (resp. t-copositive) linear maps.

Horodecki's (1996): Same result for $s = 1$
Terhal + Horodecki (2000): Same result for general cases through the the Jamiołkowski-Choi isomorphism.

$$
\begin{array}{ccccccccc}
\mathbb{V}_1 & \subset & \mathbb{V}_2 & \subset & \cdots & \subset & \mathbb{V}_{m\wedge n} & = & (M_m \otimes M_n)^+ \\
\updownarrow & & \updownarrow & & & & \updownarrow & & \\
\mathbb{P}_1 & \supset & \mathbb{P}_2 & \supset & \cdots & \supset & \mathbb{P}_{m\wedge n} & \cong & (M_m \otimes M_n)^+
\end{array}
$$

Therefore, $A \notin \mathbb{V}_1$ if and only if there exists a positive map ϕ with $\langle A, \phi \rangle < 0$. In this sense, every entanglement is detected by a positive linear map.

We also have

$$
\begin{aligned}
\phi \in \mathbb{D} &\iff \langle A, \phi \rangle \geq 0 \text{ for each } A \in \mathbb{T}, \\
A \in \mathbb{T} &\iff \langle A, \phi \rangle \geq 0 \text{ for each } \phi \in \mathbb{D},
\end{aligned}
$$

where $\mathbb{D}$ denotes the cone of all decomposable positive maps.

A self-adjoint matrix $W \in M_m \otimes M_n$ is an *entanglement witness* (Terhal, 2000) if

1. $\mathrm{Tr}\,(WA) \geq 0$ for each $A \in \mathbb{V}_1$,
2. $\mathrm{Tr}\,(WA_0) < 0$ for some $A_0 \in \mathbb{V}_{m\wedge n} = (M_m \otimes M_n)^+$.

If we consider $W = C_\phi^{\mathrm{t}}$ then the first condition says that ϕ is positive, and the second condition says that ϕ is not $m \wedge n$-positive, that is, not completely positive.

Therefore, an entanglement witness is a positive linear map which is not completely positive.

An entanglement witness is *optimal* (Lewenstein, Kraus, P. Horodecki and Cirac, 2000) if it detect a maximal set of entanglement with respect to the set inclusion.

It is easy to see that a positive map ϕ is an optimal entanglement witness if and only if the smallest face $\mathbb{P}_\phi$ of the cone $\mathbb{P}_1$ determined by ϕ does not contain a completely positive map.

The whole facial structures of the cone $\mathbb{P}_1$ is far from being understood. For the case of $m = n = 2$, Sørmer (1963) found all extremal points of the convex set of all unital positive maps, and Byeon + K (2002) characterize the whole facial structures in terms of pairs of subspaces of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

An extremal positive map which is not completely positive map is a natural example of an optimal entanglement witness.

Actually, The set of all exposed positive maps are enough to detect entanglement completely, since the set of all exposed points is dense in the set of all extreme points.

A completely copositive map $\phi^W : X \mapsto W^* X^{\rm t} W$ generates an exposed ray of the cone $\mathbb{P}_1$ (Yopp+Hill, 2005, Marciniak, 2011). If $\operatorname{rank} W \geq 2$ then this map is not completely positive. These are all exposed decomposable maps which are not completely positive.

The Choi (1977) gave an example of indecomposable extremal positive map. But, concrete examples of exposed indecomposable positive maps became to be known very recently by works by K.-C. Ha + K and Chruściński.

Another sufficient condition for optimality is the *spanning property* (Lewenstein, Kraus, P. Horodecki and Cirac, 2000). A positive map ϕ has the spanning property if the set

$$\{\bar{x} \otimes y : \langle (x \otimes y)(x \otimes y)^*, \phi \rangle = (\phi(xx^*)y \,|\, y) = 0\}$$

spans the whole space $\mathbb{C}^m \otimes \mathbb{C}^n$. Note that the product vectors $x \otimes y$ themselves with the condition never span the whole space.

It turns out that ϕ has the spanning property if and only if the smallest *exposed* face of the cone $\mathbb{P}_1$ determined by ϕ has no completely positive map. Note that the Choi map has not the spanning property.

extreme

$\nearrow$ $\searrow$

exposed optimal

$\searrow$ $\nearrow$

spanning property

3. Optimality: Decomposable cases

It is known (Lewenstein, Kraus, P. Horodecki and Cirac, 2000) that if a decomposable positive map is an optimal entanglement witness then it is completely copositive of the form

$$\phi^{\mathcal{W}} : X \mapsto W_1^* X^{\mathrm{t}} W_1 + W_2^* X^{\mathrm{t}} W_2 + \cdots W_r^* X^{\mathrm{t}} W_r,$$

where $\mathrm{span}\,\mathcal{W} = \mathrm{span}\,\{W_1, W_2, \dots W_r\}$ is completely entangled subspace of $\mathbb{C}^m \otimes \mathbb{C}^n$, that is $\mathcal{W}$ has no product vector.

1. $\Phi^{\mathcal{W}}$ has the spanning property.
2. $\Phi^{\mathcal{W}}$ is an optimal entanglement witness.
3. $\mathrm{span}\,\mathcal{W}$ is completely entangled.

Question: Is the converse (3) $\Longrightarrow$ (1) true?

This is equivalent to ask the following:

Let D be a completely entangled subspace of $\mathbb{C}^m \otimes \mathbb{C}^n$. Do there exist product vectors $x_1 \otimes y_1, x_2 \otimes y_2, \dots x_\nu \otimes y_\nu$ such that

$$\begin{cases} D^\perp & = \mathrm{span}\,\{x_1 \otimes y_1, \dots x_\nu \otimes y_\nu\}, \\ \mathbb{C}^m \otimes \mathbb{C}^n & = \mathrm{span}\,\{\bar{x}_1 \otimes y_1, \dots \bar{x}_\nu \otimes y_\nu\}. \end{cases}$$

Yes, when $m = 2$ (Augusiak, Tura and Lewenstein, 2011)

Examples of completely entangled subspaces with maximal dimensions:

$$e_{1,1} + e_{2,2}, \quad e_{1,2} + e_{2,3}, \quad e_{1,3} + e_{2,4}, \quad \dots, \quad e_{1,n-1} + e_{2,n}$$

Another necessary conditions for the optimality:

Theorem

If $\phi_{\mathcal{W}}$ is an optimal entanglement witness then we have

1. $\mathrm{span}\,\mathcal{W}$ *is completely entangled.*
2. *The orthogonal complement of* $\mathrm{span}\,\mathcal{W}$ *has a product vector.*
3. *The convex hull of* $\{\phi^{W} : W \in \mathrm{span}\,\mathcal{W}\}$ *is a face of the cone* $\mathbb{D}$.

It is well-known that the maximal dimension of completely entangled subspace of $\mathbb{C}^m \otimes \mathbb{C}^n$ is given by $(m-1)(n-1)$.

In the case of $m=2$, this is $n-1$.
in the case of $m=n=3$, this is $2 \times 2 = 4$. Therefore, the condition (1) automatically implies the condition (2). In this case, we have an example of 4-dimensional CES which violated the condition (3). This is spanned by

$$\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}, \quad \begin{pmatrix} \cdot & b & \cdot \\ \frac{1}{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & b \\ \cdot & \frac{1}{b} & \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot & \cdot & \frac{1}{b} \\ \cdot & \cdot & \cdot \\ b & \cdot & \cdot \end{pmatrix}$$

This is the support of a variant of an example of PPTES given by Choi (1980)

In the case of $m = 3$ and $n = 4$, the maximal dimension of CES is $6 = 12 - 6$. It was informed by Young-Hoon Kiem that if

$$mn - (m-1)(n-1) \leq k \leq (m-1)(n-1)$$

then k-dimensional subspaces of $M_{m \times n}$ are generically completely entangled subspaces with the completely entangled orthogonal complements.

Explicit such examples for $m = 3$ and $n = 4$ were given recently by Remigiusz Augusiak and Łukasz Skowronek independently.

Question: What about the converse of Theorem.

4. Optimality: Indecomposable cases

By the relation

$$\phi \in \mathbb{D} \iff \langle A, \phi \rangle \geq 0 \text{ for each } A \in \mathbb{T},$$
$$A \in \mathbb{T} \iff \langle A, \phi \rangle \geq 0 \text{ for each } \phi \in \mathbb{D},$$

we see that an entanglement witness ϕ detects a PPTES if and only if ϕ is indecomposable. An entanglement witness is said to be *non-decomposable optimal entanglement witness (nd-OEW)* if it detects a maximal set of PPTES. ((Lewenstein, Kraus, P. Horodecki and Cirac, 2000))

Question: If ϕ is non-decomposable and an optimal entanglement witness then is ϕ really nd-OEW in the above sense?

No.

It is easy to see that an indecomposable map ϕ is an nd-OEW in the above sense if and only if both ϕ and the composition $\phi \circ \mathrm{t}$ by the transpose map are optimal. In the isomorphism $\phi \leftrightarrow C_\phi^{\mathrm{t}}$, this is equivalent to say that both a self-adjoint matrix $W = C_\phi^{\mathrm{t}}$ and its partial transpose W^τ are optimal.

We say that A positive linear map ϕ is said to

- be *co-optimal* if the smallest face of $\mathbb{P}_1$ containing ϕ has no completely copositive map.
- be *bi-optimal* if it is optimal and co-optimal.
- have the *co-spanning property* if the smallest exposed face of $\mathbb{P}_1$ containing ϕ has no completely copositive map.
- have the *bi-spanning property* if it has both the spanning and co-spanning property.

ϕ is co-optimal if and only if $\phi \circ \mathrm{t}$ is optimal, and similarly for co-spanning property.
These properties depends on faces: two interior point of a face share properties.

We test the above properties for the linear map $\Phi[a,b,c] : M_3 \to M_3$ which sends $[x_{ij}] \in M_3$ to

$$\begin{pmatrix} ax_{11}+bx_{22}+cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11}+ax_{22}+bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11}+cx_{22}+ax_{33} \end{pmatrix},$$

where a, b and c are nonnegative real numbers.

The corresponding Choi matrix is given by

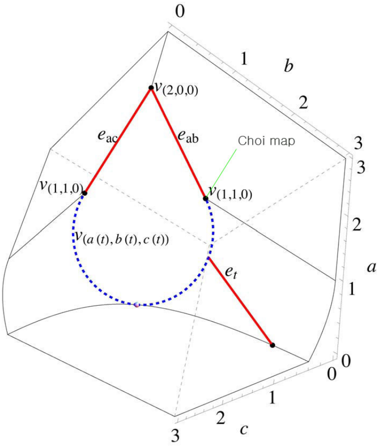
$$W[a,b,c] = \begin{pmatrix} a & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & a \end{pmatrix}.$$

- $\Phi[1,2,2]$: a 2-positive linear map which is not completely positive (Choi, 1972).
- $\Phi[1,0,\mu]$ with $\mu \geq 1$: the first example of an indecomposable positive linear map (Choi, 1975)
- $\Phi[1,0,1]$: extremal (Choi + Lam, 1977)
- $\Phi[1,0,1]$: is not the sum of a 2-positive map and a 2-copositive map (Tanahashi and J. Tomiyama, 1988)
- $\Phi[1,0,1]$: has not the spanning property (K, 1996)
- $\Phi[1,0,1]$: has the co-spanning property (H.-S. Choi and K, 2012)

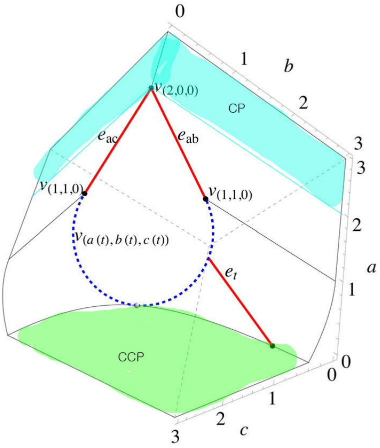
The map $\Phi[a, b, c]$ is positive if and only if the condition

$$a + b + c \geq 2, \qquad 0 \leq a \leq 1 \Longrightarrow bc \geq (1 - a)^2$$

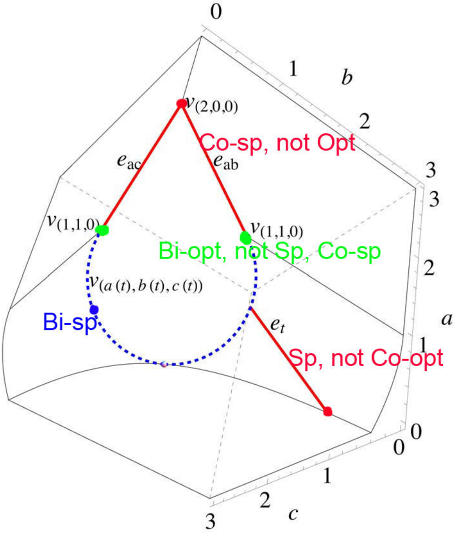
holds. (Cho + K + Lee, 1990)



$\Phi[a, b, c]$ is completely positive if and only if $a \geq 2$
$\Phi[a, b, c]$ is completely copositive if and only if $bc \geq 1$



Results with K.-C. Ha:



We have found examples of entanglement witnesses with

- the spanning property, not co-optimal, not extremal: These give examples of optimal, indecomposable EW's, which are not nd-OEW in the current terminology. These also show that the spanning property does not imply the extremeness. Note that the Choi map show that extremeness does not imply the spanning property.
- the co-spanning property, not optimal, not extreme: the composition with the transpose map give the same kinds of examples.

Recently, Ha + Yu found examples of bi-optimal EW's which are neither extremal nor have the spanning property.

Further result (with K.-C. Ha)

We have consider the linear map in M_3 whose Choi matrix is given by

$$\begin{pmatrix} a & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} \\ \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ -e^{-i\theta} & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & -e^{i\theta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot \\ -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & a \end{pmatrix}.$$

- $\theta = 0$: positive $\neq$ decomposable (Choi); PPT $=$ separable
- $\theta = \pi$: positive $=$ decomposable; PPT $\neq$ separable (Størmer)

Parameterize 1 and -1 with $e^{i\theta}$.

- Characterize positivity of the maps.
- Find optimal entanglement witnesses detecting the PPT edge states of type $(6, 8)$ by K+ H. Osaka.
- Disprove the SPA (structural physical approximation) conjecture (J. K. Korbicz, M. L. Almeida, J. Bae, M. Lewenstein and A. Acin, 2008).

感謝