Sofic entropy via finite partitions

David Kerr

Texas A&M University

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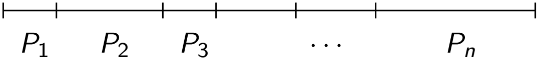
Shannon entropy

The *Shannon entropy* of a partition $\mathcal{P}$



of a probability space (X, μ) is defined as

$$H(\mathcal{P}) = -\sum_{i=1}^{n} \mu(P_i) \log \mu(P_i),$$

which can be viewed as the integral of the information function

$$I(x) = -\log \mu(P_i)$$

where i is such that $x \in P_i$.

Kolmogorov-Sinai entropy

For a single measure-preserving transformation $T : X \to X$ we set

$$h_\mu(T, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H(\mathcal{P} \vee T^{-1}\mathcal{P} \vee \cdots \vee T^{-n+1}\mathcal{P})$$
$$h_\mu(T) = \sup_{\mathcal{P}} h_\mu(T, \mathcal{P}).$$

Kolmogorov-Sinai theorem

If $\mathcal{P}$ is a finite generating partition then $h_\mu(T) = h_\mu(T, \mathcal{P})$.

As a consequence, the entropy of the Bernoulli shift on $(X_0, \mu_0)^{\mathbb{Z}}$ is equal to the Shannon entropy of the base.

Theorem (Ornstein)

Bernoulli shifts are classified by their entropy.

This theory applies most generally to **amenable** acting groups.

Bowen's measure entropy

Basic idea

Replace internal averaging (information theory) by the counting of discrete models (statistical mechanics).

Let $\mathcal{P}$ be a partition of X whose atoms have measures $c_1, \ldots, c_n$. In how many ways can we approximately model this ordered distribution of measures by a partition of $\{1, \ldots, m\}$ for a given $m \in \mathbb{N}$? By Stirling's formula, the number of models is roughly

$$c_1^{-c_1 m} \cdots c_n^{-c_n m}$$

for large m, so that

$$\frac{1}{m} \log(\#\text{models}) \approx -\sum_{i=1}^{n} c_i \log c_i = H(\mathcal{P}).$$

Bowen's measure entropy

Let $G \curvearrowright (X, \mu)$ be a measure-preserving action, and let Σ be a sequence of maps $\sigma_i : G \to \mathrm{Sym}(m_i)$ into finite permutation groups which are asymptotically multiplicative and free (the existence of such a sequence defines a **sofic** group).

Let $\mathcal{P}$ be a finite ordered partition of X. For a finite set $F \subseteq G$ and $\varepsilon > 0$ we write $\mathrm{AP}(\mathcal{P}, F, \varepsilon, \sigma_i)$ for the number of ordered partitions $\mathcal{Q}$ of $\{1, \ldots, m_i\}$ such that the measures of the atoms of $\bigvee_{s \in F} s^{-1}\mathcal{P}$ and $\bigvee_{s \in F} s^{-1}\mathcal{Q}$ which correspond to each other under the dynamics are summably ε-close. Set

$$h_{\Sigma,\mu}(\mathcal{P}) = \inf_F \inf_{\varepsilon > 0} \limsup_{i \to \infty} \frac{1}{m_i} \log \#\mathrm{AP}(\mathcal{P}, F, \varepsilon, \sigma_i)$$

Theorem (Bowen)

$h_{\Sigma,\mu}(\mathcal{P})$ has a common value for generating partitions $\mathcal{P}$.

Linear reformulation

On the set of unital positive maps $L^\infty(X, \mu) \to \mathbb{C}^{m_i}$ we define the pseudometric

$$\rho_{\mathcal{P}}(\varphi, \psi) = \max_{f \in \mathcal{P}} \|\varphi(f) - \psi(f)\|_2.$$

For $\delta > 0$ define $\mathrm{UP}_\mu(\mathcal{P}, F, \delta, \sigma_i)$ to be the set of all unital positive maps $L^\infty(X, \mu) \to \mathbb{C}^{m_i}$ which, to within δ, are approximately multiplicative and F-equivariant and approximately pull the uniform probability measure on $\{1, \ldots, m_i\}$ back to μ.

Proposition

$$h_{\Sigma,\mu}(\mathcal{P}) = \sup_{\varepsilon>0} \inf_{F} \inf_{\delta>0} \limsup_{i\to\infty} \frac{1}{m_i} \log N_\varepsilon(\mathrm{UP}_\mu(\mathcal{P}, F, \delta, \sigma_i))$$

where $N_\varepsilon(\cdot)$ denotes the maximal cardinality of an ε-separated set.

Linear reformulation

The previous proposition can furthermore be used as a definition of $h_{\Sigma,\mu}(\mathcal{P})$ when $\mathcal{P}$ is any finite subset of $L^\infty(X,\mu)$. One can also more generally define $h_{\Sigma,\mu}(\mathcal{S})$ for any bounded sequence $\mathcal{S}$ in $L^\infty(X,\mu)$. We then have the following.

Theorem (K.-Li)

$h_{\Sigma,\mu}(\mathcal{S})$ has a common value over all dynamically generating bounded sequences $\mathcal{S}$ in $L^\infty(X,\mu)$.

Definition

The measure entropy $h_{\Sigma,\mu}(X,G)$ of the action $G \curvearrowright X$ is defined as the common value in the above theorem.

Linear reformulation

The topological entropy $h_\Sigma(X, G)$ of an action of G on a compact metrizable space X can be defined similarly. It measures the exponential growth of the number of approximately equivariant maps $\{1, \dots, m_i\} \to X$ that can be distinguished up to some error.

Theorem (variational principle)

Let $G \curvearrowright X$ be an action on a compact metrizable space. Then

$$h_\Sigma(X, G) = \sup_\mu h_{\Sigma,\mu}(X, G)$$

where μ ranges over all invariant Borel probability measures on X.

The sofic topological and measure entropies coincide with their classical counterparts when G is amenable, and so this extends the classical variational principle.

A generator-free definition of sofic entropy

We seek a general generator-free definition of sofic entropy in the spirit of what Sinai furnished for single transformations in response to Kolmogorov's generator-based definition.

Basic idea

Measure the exponential growth of the number of sofic models as before but for **each** *partition at some* **fixed observational scale**, *and then take a supremum of these growth rates as the scale becomes finer and finer.*

The observational scale is determined by a second partition, and so the parameters in the definition now include two partitions playing different roles.

A generator-free definition of sofic entropy

Define $\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)$ to be the set of all homomorphisms from the algebra generated by $\mathcal{P}$ to the algebra of subsets of $\{1, \dots, m_i\}$ which, to within δ,

- are approximately F-equivariant, and
- approximately pull back the uniform probability measure on $\{1, \dots, m_i\}$ to μ.

For a partition $\mathcal{Q} \leq \mathcal{P}$, write $|\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)|_\mathcal{Q}$ for the cardinality of the set of restrictions of elements of $\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)$ to $\mathcal{Q}$.

Definition

$$h_{\Sigma,\mu}(X, G) = \sup_{\mathcal{Q}} \inf_{\mathcal{P} \geq \mathcal{Q}} \inf_{F,\delta} \limsup_{i \to \infty} \frac{1}{m_i} \log |\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)|_\mathcal{Q}$$

A generator-free definition of sofic entropy

We then have the following Kolmogorov-Sinai-type theorem, which enables us to compute the entropy as we have defined it.

Theorem

In the definition of $h_{\Sigma,\mu}(X, G)$, one can equivalently restrict the partitions $\mathcal{P}$ and $\mathcal{Q}$ to range within a given generating σ-algebra. In particular, if there is a finite generating partition then $\mathcal{P}$ and $\mathcal{Q}$ need not range over any partitions except this one.

The above theorem permits us to show that our definition is equivalent to Bowen's in the presence of a generating partition.

Bernoulli actions

For a probability space (Y, ν) write $H(\nu)$ for the supremum of $H_\nu(\mathcal{Q})$ over all finite partitions $\mathcal{Q}$ of Y.

Theorem

Let (Y, ν) be a probability space and let $G \curvearrowright (Y^G, \nu^G)$ be the associated Bernoulli action. Then

$$h_{\Sigma,\nu^G}(Y^G, G) = H(\nu).$$

Proof

Let $\mathcal{Q}$ be a finite partition consisting of cylinder sets over e. The collection of such partitions is generating for the action. The entropy with respect to $\mathcal{Q}$ is easily seen to be bounded above by $H_\mu(\mathcal{Q})$, so that $h_{\Sigma,\nu^G}(Y^G, G) \leq H(\nu)$.

Bernoulli actions

For the reverse inequality it suffices to show, by the monotonicity properties of entropy, that

$$\inf_{F,\delta} \limsup_{i\to\infty} \frac{1}{m_i} \log |\mathrm{Hom}_\mu(\mathcal{Q}, F, \delta, \sigma_i)|_{\mathcal{Q}} \geq H_\mu(\mathcal{Q}).$$

To do this, we enumerate the elements of $\mathcal{P}$ as $A_1, \dots, A_n$ and think of homomorphisms from the algebra generated by $\mathcal{Q}$ to the algebra of subsets of $\{1, \dots, m_i\}$ as elements of $\{1, \dots, n\}^{m_i}$, which we regard as a probability space under the measure ν^{m_i}.

The inequality then ensues by combining two observations:

1. Using Chebyshev's inequality, one shows that a random element of $\{1, \dots, n\}^{m_i}$ with high probability gives a homomorphism in $\mathrm{Hom}_\mu(\mathcal{Q}, F, \delta, \sigma_i)$ for prescribed F and δ.

2. The law of large numbers yields

$$\lim_{m_i \to \infty} \mathbf{P}\Big(\Big| -\frac{1}{m_i} \log \nu^{m_i}(\gamma) - H(\nu) \Big| > \delta \Big) = 0,$$

   so that for large m_i there is an $L \subseteq \{1, \dots, n\}^{m_i}$ for which $\nu^{m_i}(L) > 1 - \delta$ and

$$\nu^{m_i}(\{\gamma\}) \le e^{-m_i(H(\nu)-\delta)}$$

   for all $\gamma \in L$.

Bernoulli actions

Theorem

Bernoulli actions of countable sofic groups have **completely positive entropy***, which means that every nontrivial factor has strictly positive entropy with respect to every sofic approximation sequence.*

Bowen's f-invariant

Let $F_r \curvearrowright (X, \mu)$ be a measure-preserving action of a free group on r generators $s_1, \ldots, s_r$. Write B_n for the set of words in $s_1, \ldots, s_r$ of length at most n. For a finite partition $\mathcal{P}$ of X set

$$F(\mathcal{P}) = (1 - 2r)H(\mathcal{P}) + \sum_{i=1}^{r} H(\mathcal{P} \vee s_i^{-1}\mathcal{P}),$$

$$f(\mathcal{P}) = \inf_{n \in \mathbb{N}} F\Big(\bigvee_{s \in B_n} s^{-1}\mathcal{P}\Big)$$

This last quantity is the same for all generating partitions $\mathcal{P}$, and in the case that there exists a generating partition we define the f-**invariant** of the action to be this common value.

Bowen's f-invariant

Bowen showed that the f-invariant coincides with a version of sofic entropy which is locally computed by **averaging over all sofic approximations** on a finite set instead of using a given sofic approximation.

Corollary

Every nontrivial factor of a Bernoulli action of F_r possessing a finite generating partition has strictly positive f-invariant.