Are all AW^*-algebras monotone complete?

Masayoshi Kaneda
Nazarbayev University
Astana, Republic of Kazakhstan

Operator Spaces, Quantum Probability and Applications
Wuhan University, People's Republic of China
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History of AW^*-algebras

- Kaplansky defined AW^*-algebras in an attempt to give an abstract algebraic characterization of von Neumann algebras (W^*-algebras) in his paper published in the *Annals of Mathematics* in 1951. Hence comes the name AW^*-algebras (i.e., Abstract W^*-algebras).

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- In 1951, Dixmier found that there are commutative AW^*-algebras which are not von Neumann algebras.
- Since then, more attention has been paid to monotone complete C^*-algebras which are more maneuverable than AW^*-algebras.
- The question "Are all AW^*-algebras monotone complete?" arouse.

Definitions

- A C^*-algebra $\mathcal{A}$ is *monotone complete* if and only if every norm-bounded monotone increasing net in $\mathcal{A}_{sa}$ has a least upper bound in $\mathcal{A}_{sa}$.

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- A C^*-algebra $\mathcal{A}$ is *monotone complete* if and only if every norm-bounded monotone increasing net in $\mathcal{A}_{sa}$ has a least upper bound in $\mathcal{A}_{sa}$.
- A C^*-algebra $\mathcal{A}$ is an *AW^*-algebra* if and only if every maximal abelian C^*-subalgebra (MASA) is monotone complete.

Remark

Remark: It is straightforward to see that every monotone complete C^*-algebra $\mathcal{A}$ is an AW^*-algebra. The following argument is due to J. D. Maitland Wright. Let $\mathcal{C}$ be a MASA of $\mathcal{A}$ and $\{x_\alpha\}$ a norm-bounded monotone increasing net in $\mathcal{C}_{sa}$. Then $\{x_\alpha\}$ has a least upper bound x in $\mathcal{A}_{sa}$. To show that x is actually in $\mathcal{C}_{sa}$, let u be any unitary in $\mathcal{C}$. Then $u^*xu \geq u^*x_\alpha u = u^*ux_\alpha = x_\alpha, \forall\alpha$. Thus $u^*xu \geq x$. Similarly, by replacing u by u^*, we have that $uxu^* \geq x$, and so $x \geq u^*xu$. Hence $u^*xu = x$, so that $xu = ux$. Since u is any unitary in $\mathcal{C}$, x commutes any element of $\mathcal{C}$. By maximality of $\mathcal{C}$, $x \in \mathcal{C}$.

Conjecture and Idea of Proof

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- Theorem (Pedersen 1972)

  If every MASA of a concrete C^*-algebra $\mathcal{A}$ is weakly closed, then $\mathcal{A}$ is weakly closed.

Proof

- **Proof (with One Gap!)**: Let $\mathcal{A}$ be an AW^*-algebra nondegenerate on a Hilbert space $\mathcal{H}$ (hence $\mathcal{A}$ contains the identity operator $1_{\mathcal{H}}$ on $\mathcal{H}$), and let ϕ be a minimal completely positive idempotent on $\mathbb{B}(\mathcal{H})$ that fixes each element of $\mathcal{A}$. Then the image $\mathrm{Im}\phi$ is an injective envelope of $\mathcal{A}$, and hence it is denoted by $I(\mathcal{A})$. Define

$$S := \{x \in \mathbb{B}(\mathcal{H})_{sa} : \phi(x) \in \mathcal{A}, \phi(x^2) = \phi(x)^2\}.$$

Clearly S is a norm-closed subset of $\mathbb{B}(\mathcal{H})_{sa}$ containing $\mathcal{A}_{sa}$, and $\phi(S) = \mathcal{A}_{sa}$.

Proof

- Claim 1: If $\{x_k\}_{k=1}^{n} \subset S$, then $\phi(x_1 \cdots x_n) = \phi(x_1) \cdots \phi(x_n)$.

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- $\because$ By Choi's multiplicative domain theorem for a 2-positive mapping ϕ,

$$\begin{aligned} &\{a \in \mathbb{B}(\mathcal{H})_{sa} : \phi(a^2) = \phi(a)^2\} \\ =&\{a \in \mathbb{B}(\mathcal{H})_{sa} : \phi(ba) = \phi(b)\phi(a), \forall b \in \mathbb{B}(\mathcal{H})\}. \end{aligned}$$

  Since $\phi(x_n^2) = \phi(x_n)^2$, applying this theorem with $b = x_1 \cdots x_{n-1}$ and $a = x_n$ yields that $\phi(x_1 \cdots x_n) = \phi(x_1 \cdots x_{n-1})\phi(x_n)$. Now the assertion follows by mathematical induction.

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- $\because$ Let $x, y \in S$ and $t \in \mathbb{R}$, then $\phi(tx + y) = t\phi(x) + \phi(y) \in \mathcal{A}$. That $\phi((tx + y)^2) = \phi(tx + y)^2$ follows from Claim 1.

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- Define $V := S + iS$, then $\phi(V) = \mathcal{A}$.

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- Claim 3: If $z \in V$, then $z^* z \in S$.

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- Define $V := S + iS$, then $\phi(V) = \mathcal{A}$.
- Claim 3: If $z \in V$, then $z^*z \in S$.
- $\because$ Express z as $z = x + iy$, where $x, y \in S$. Then by Claim 1, $\phi(z^*z) = \phi((x + iy)^*(x + iy)) = (\phi(x) + i\phi(y))^*(\phi(x) + i\phi(y)) \in \mathcal{A}$ and $\phi((z^*z)^2) = \phi(((x + iy)^*(x + iy))^2) = \phi((x + iy)^*(x + iy))^2 = \phi(z^*z)^2$, and hence $z^*z \in S$.

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- Claim 4: If $x, y \in S$, then $yx \in V$.
- $\because$ This follows from the polarization identity $yx = \frac{1}{4} \sum_{k=0}^{3} i^k (x + i^k y)^* (x + i^k y)$ and Claim 3.
- Therefore, V is a C^*-algebra, and by Claim 1 ϕ is a $*$-epimorphism from V onto $\mathcal{A}$.

Proof

- Claim 5: If $\{x_\alpha\}$ is a norm-bounded monotone increasing net of pairwise commuting elements from S strongly converging to $x \in \mathbb{B}(\mathcal{H})$, then $x \in S$.

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- $\because$ $\{\phi(x_\alpha)\}$ is pairwise commuting since $\{x_\alpha\}$ is so and ϕ is multiplicative. Let $\mathcal{C}$ be a MASA containing $\{\phi(x_\alpha)\}$. Since $\mathcal{A}$ is an AW^*-algebra, $\{\phi(x_\alpha)\}$ has a least upper bound y in $\mathcal{C}_{sa}$. On the other hand, x is clearly the least upper bound of $\{x_\alpha\}$ in $\mathbb{B}(\mathcal{H})_{sa}$, and hence $\phi(x)$ is an upper bound of $\{\phi(x_\alpha)\}$ in $I(\mathcal{A})_{sa}$. Assume that $\phi(x) = y$. (This is the gap!) Then $\phi(x) = y \in \mathcal{A}$.

Proof

To see that $\phi(x^2) = \phi(x)^2$, first note that $\{x_\alpha^2\} \subset S$ by Claim 3, and that $\{x_\alpha^2\}$ is pairwise commuting and monotone increasing (for $x_\alpha \leq x_\beta$ implies that $x_\alpha^2 \leq x_\beta^2$ since $\{x_\alpha\}$ is pairwise commuting) and strongly converging to x^2. Thus as in the former part of the proof of the present claim, $\phi(x^2)$ is the least upper bound of $\{\phi(x_\alpha^2)\}(= \{\phi(x_\alpha)^2\})$ in $\mathcal{C}_{sa}$, and $\phi(x)$ commutes with each $\phi(x_\alpha)$ by Claim 1 and the fact that x clearly commutes with each x_α. Thus $\phi(x_\alpha) \leq \phi(x)$ implies that $\phi(x_\alpha)^2 \leq \phi(x)^2$ and hence $\phi(x^2) \leq \phi(x)^2$. Together with the Kadison-Schwarz Inequality for 2-positive mapping $\phi(x)^2 \leq \phi(x^2)$, we have that $\phi(x^2) = \phi(x)^2$. Therefore $x \in S$, and Claim 5 has been shown.

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- Lemma (Kadison 1956)

  A concrete C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a von Neumann algebra if and only if $(\mathcal{A}_{sa})^m = \mathcal{A}_{sa}$, where $(\mathcal{A}_{sa})^m$ denotes the set of elements in $\mathbb{B}(\mathcal{H})$ which can be obtained as strong limits of monotone increasing nets from $\mathcal{A}_{sa}$.

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  - By Claim 5 together with Kadison's lemma, every MASA of V is weakly closed.
  - By Pedersen's theorem, V is weakly closed, i.e., V is a von Neumann algebra.

Proof

- Thus, $\mathcal{A}$ is the image of a von Neumann algebra V by the $*$-epimorphism ϕ which is also an idempotent.

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- Thus, $\mathcal{A}$ is the image of a von Neumann algebra V by the $*$-epimorphism ϕ which is also an idempotent.
- Now it is clear that $\mathcal{A}$ is monotone complete. Indeed, if $\{x_\alpha\}$ is a norm-bounded monotone increasing net in $\mathcal{A}_{sa}$, then by Vigier's theorem it strongly converges to some $x \in V$. Then $\phi(x) \in \mathcal{A}_{sa}$ serves as the least upper bound of $\{x_\alpha\}$ in $\mathcal{A}_{sa}$.

Corollaries

- Corollary (Conjecture)

  For a C^*-algebra $\mathcal{A}$, the following are equivalent:

  i) *$\mathcal{A}$ is an AW^*-algebra;*
  ii) *$\mathcal{A}$ is a monotone complete C^*-algebra;*
  iii) *$\mathcal{A}$ is a quotient of a von Neumann algebra. More specifically, $\mathcal{A}$ is faithfully represented on a Hilbert space as the image of a von Neumann algebra by a unital completely positive idempotent ϕ.*

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- Corollary (Conjecture)

  Every AW^*-algebra is the norm closure of the linear span of its projections.

Open Questions Which May Fill the Gap

- **Question 1**: Suppose that $1_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B} \subset \mathbb{B}(\mathcal{H})$ be a sequence of C^*-subalgebras with $\phi : \mathcal{B} \to \mathcal{A}$ a $*$-epimorphism such that $\phi^2 = \phi$ and $\mathcal{A}$ an AW^*-algebra nondegenerate on $\mathcal{H}$. For each norm-bounded monotone increasing pairwise commuting net $\{x_\alpha\}$ with strong limit $x \in \mathbb{B}(\mathcal{H})$, does ϕ extend to a 2-positive mapping from span$(\mathcal{B} \cup \{x\})$ onto A?

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- The affirmative answer to this question would be sufficient to conclude that "All AW^*-algebras are monotone complete".

Open Questions Which May Fill the Gap

- **Question 2**: Suppose that $1_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B} \subset \mathbb{B}(\mathcal{H})$ be a sequence of C^*-subalgebras with $\phi : \mathcal{B} \to \mathcal{A}$ a $*$-epimorphism such that $\phi^2 = \phi$ and $\mathcal{A}$ monotone complete and nondegenerate on $\mathcal{H}$. For a given norm-bounded monotone increasing net $\{x_\alpha\}$ with strong limit $x \in \mathbb{B}(\mathcal{H})$, let $\tilde{\phi} : \text{span}(\mathcal{B} \cup \{x\}) \to \text{span}(\mathcal{A} \cup \{z\})$ be a linear extension of ϕ such that $\tilde{\phi}(x) = z$, where z is the strong limit of $\{\phi(x_\alpha)\}$ in $\mathbb{B}(\mathcal{H})$. Is $\tilde{\phi}$ 2-positive?

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- The affirmative answer to this question would be sufficient to conclude that "Every monotone complete C^*-algebra is a quotient of a von Neumann algebra".