Harmonic analysis via cocylces

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Classical examples

- An L_p-Fourier multiplier is given by a function $m : \mathbb{R}^n \to \mathbb{C}$ such that
$$\widehat{T_m(f)}(\xi) \;=\; m(\xi)\hat{f}(\xi) \;,$$
where $\hat{f}(\xi) \;=\; \int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi}^n} e^{-i(\xi,x)} f(x)dx$ is the Fourier transform.
- Fourier multipliers are used for ODE/PDE, and omnipresent in classical analysis.
- For $p = 2$, it suffices to assume that m is bounded. For $p \in \{1, \infty\}$ we need $\hat{m} \in L_1$. For $1 < p < \infty$ there is in general no precise criterion in terms for m which characterizes L_p boundedness (except for the radial case-Seeger).

Key examples

- The Hilbert transform on $\mathbb{R}$ is given by $m(\xi) = \mathrm{sign}\xi$.
- The Hilbert transform on $L_p(\mathbb{T})$ is given by $m(k) = \mathrm{sgn}(k)$. Note here that f is defined on $\mathbb{Z}$ and M_m on $L_p(\hat{\mathbb{Z}})$.
- The directional Riesz transform on $L_p(\mathbb{T}^n)$ or $L_p(\mathbb{R}^n)$ is given by

$$f(\xi) \; = \; \frac{(\eta, \xi)}{(\xi, \xi)^{1/2}} \, .$$

- The 'full' Riesz transform is given by the Hilbert space-valued function $m(\xi) \; = \; \frac{\xi}{\|\xi\|}$, and $M_m : L_p(\mathbb{T}^n) \to L_p(\mathbb{T}^n, \ell_2^n)$.
- The boundedness Riesz transform is an important criterion when investigating properties of domains and their boundary (e.g. work of Steve Hoffmann).

Classical Mihlin multiplier theorem for $\mathbb{Z}^n$

1. Let $F : \mathbb{R}^n \to \mathbb{C}$ be differentiable such that

$$|D_\alpha F(\xi)||\xi|^\alpha \leq c_\alpha \quad \text{for all } |\alpha| \leq \lfloor \frac{n+1}{2} \rfloor .$$

   Then M_F is bounded on all $L_p(\mathbb{R}^n)$.

2. The restriction of F to $\mathbb{Z}^n$ is bounded on $L_p(\mathbb{T}^n)$.

Our Goals

- Multiplier theorems on discrete groups;
- Singular integrals on von Neumann algebras;
- Noncommutative analysis for noncommutative spaces.

Setup

- Let G be a discrete group. Then

$$\Delta(g) \;=\; g \otimes g$$

extends to co-multiplication on $A = \mathbb{C}[G]$ the group algebra, (functions with finite support!). Indeed,

$$(id \otimes \Delta)\Delta(g) \;=\; g \otimes g \otimes g \;=\; (\Delta \otimes id)\Delta(g)\;.$$

- An (algebraic) Fourier multiplier is a linear map $T : \mathbb{C}[G] \to \mathbb{C}[G]$ such that

$$(id \otimes T)\Delta \;=\; \Delta \circ T\;.$$

- **Exercise:** Then $T(\lambda(g)) = f(g)\lambda(g)$ for some f.
- Boundedness on $\ell_2(G) = L_2(\hat{G})$ iff f is bounded.

Noncommutative L_p on the dual of the discrete group

- ✍ We will use the reduced algebra $\hat{G} = LG = \lambda(G)'' \subset B(\ell_2(G))$ given by the left regular representation $\lambda(g)e_h = e_{gh}$.
- ✍ $\tau(x) = (e_1, xe_1)$ extends to a normal faithful tracial state on $\lambda(G)''$.
- ✍ $L_p(LG) = L_p(\hat{\mathbb{G}})$ is the completion of LG with respect to $\|x\|_p \; = \; [\tau(|x|^p)]^{\frac{1}{p}}$.
- ✍ Let $x = \sum_w a_w \lambda(w)$ be selfadjoint, then $\tau(x^{2k})$ is the return probability of the random walk with probabilities a_w.
- ✍ An L_p multiplier is given by a function $f : G \to \mathbb{C}$ such that

$$M_f(\lambda(g)) \; = \; f(g)\lambda(g)$$

extends to a bounded (completely) bounded multiplier on $L_p(LG)$ $(= L_p(VN(G)))$.

Abstract Singular Integrals?

- ✍ What should be an abstract singular integral, or Fourier multiplier?
- ✍ We is the analog of differentiable function serving as a substitute of restriction from $\mathbb{R}^n$ to $\mathbb{Z}^n$?
- ✍ In the theory of semigroups there are two kinds of examples:
- ✍ Let $T_t = e^{-tA}$. Then we can ask whether $f(A) : L_p \to L_p$ is bounded. This leads to H^∞ calculus ... (see e.g. J.-LeMerdy-Xu for the nc case)
- ✍ Since we are in the compact case, we may follow P.A. Meyer and ask for the boundedness of the Riesz transforms

$$\|\Gamma(x,x)^{1/2}\|_p \sim \|A^{1/2}x\|_p \ , \qquad (0.1)$$

where Γ is the gradient form (“carée du champs")

$$2\Gamma(x,y) \ = \ A(x^*)y + x^*A(y) - A(x^*y) \ .$$

- ✍ Here we will consider semigroups of Fourier multipliers.

Schoenberg's theorem

- There is a one to one correspondance between
  1. A semigroup of positive definite real functions $\varphi_t : G \to \mathbb{R}$;
  2. A semigroup of completely positive maps $T_t : LG \to LG$ of trace preserving, completely positive selfadjoint maps such that $T_t(\lambda(g)) = \varphi_t(g)\lambda(g)$ admits $\lambda(g)$ as eigenfunctions.
  3. A real Hilbert space H, with a cocycle (α, b) given by a representation $\alpha : G \to O(H)$ and

$$b(gh) \ = \ \alpha_g(b(h)) + b(g) \ .$$

  Link $\varphi_t(g) = e^{-t\psi(g)} \ = \ e^{-t\|b(g)\|^2}$. ψ is a so-called conditionally negative function.
- Model: $G = \mathbb{Z}^n$, $b(g) = g \in \mathbb{R}^n$ with heat semigroup

$$T_t(\lambda(k)) \ = \ e^{-\|k\|^2}\lambda(k)$$

Riesz transforms for discrete groups

Theorem: (J.-Mei-Parcet) Let G be a discrete group and $T_t = e^{-tA}$ be a semigroup of selfadjoint multipliers and $p \geq 2$. Then

$$\|\Gamma(x,x)^{1/2}\|_p + \|\Gamma(x^*,x^*)^{1/2}\|_p \sim_{c(p)} \|A^{1/2}x\|_p \ .$$

There is a suitable formula with a sum for $1 < p \leq 2$.

Key Formula: Let $b : G \to H$ be the cocycle and think $H = \mathbb{R}^n$. Then α induces a measure preserving transformation on $L_\infty(\mathbb{R}^n)$. Then $\pi : LG \to L_\infty(\mathbb{R}^n) \rtimes G$ given by

$$\pi(\lambda(g)) \ = \ e^{i(b(g),\ )}\lambda(g)$$

satisfies

$$\pi(T_t(x)) \ = \ (T_t^{\Delta} \rtimes id_G)\pi(x) \ .$$

Here T_t^{Δ} is the classical heat semigroup on $\mathbb{R}^n$ and $T_t^{\Delta} \otimes id_{B(\ell_2(G))}$ leaves the crossed product invariant, and is denoted by $T_t^{\Delta} \rtimes id_G$.

Idea of proof

- ✏ Think $H=\mathbb{R}^n$ and apply Pisier's method for Riesz transforms on $\mathbb{R}^n$.
- ✏ The missing ingredient is a Khintchine formula for group actions, namely let $B: H \to \Gamma_1(H)$, $B(h)=\sum_k h_k G_k$ be the gaussian functor which assigns gaussian variables to Hilbert space vectors (G_k iid normal). Assume that a discrete group G acts on H. Let $x=\sum_g B(h_g)\lambda(g)$ and $p \ \geq \ 2$. Then

$$\|x\|_{L_p(L_\infty(H)\rtimes G)} \sim_{c(p)} \|E_{LG}(x^*x)^{1/2}\|_p + \|E_{LG}(xx^*)^{1/2}\|_p \, .$$

- ✏ The upper estimate holds with $c(p) \leq c\sqrt{p}$ (Junge-Zeng).
- ✏ A finite dimensional real Banach space X embeds isometrically in L_1 iff the norm is conditionally negative. The corresponding gradient form usually admits an infinite dimensional cocycle with non-trivial action ("more exotic multipliers", also by bounded cocycles).

Abstract tools

- ✐ (J.-R.-S.) Let T_t be a semigroup of unital completely positive selfadjoint maps on a von Neumann algebra N. Then T_t admits a Markov dilation, i.e. $\pi_t : N \to \mathcal{M}_t \subset \mathcal{M}$, $(\mathcal{M}_t)$ filtration contained in finite von Neumann algebra $\mathcal{M}$ such that

$$E_s\pi_t(x) \; = \; \pi_s(T_{t-s}x) \quad s < t \; .$$

- ✐ (J.-R.-S.) If in addition $\Gamma(x,x) \in L_1$ for the $*$-algebra $B = \mathrm{dom}(A^{1/2})$, then T_t admits a free brownian motion the corresponding martingales have almost uniformly continuous path.
- ✐ **Corollary:** A admits H^∞ calculus.

Interpolation

(J.-Mei) If T_t admits a Markov dilation with almost uniformly continuous path, then

$$[bmo, L_p^0(N)]_{\frac{1}{q}} \;=\; L_{pq}^0(N)$$

holds for the space of mean 0 operators and the intrinsic bmo-norm $\|x\|_{bmo} \;=\; \max\{\|x\|_{bmo_c}, \|x^*\|_{bmo_c}\}$ given by

$$\|x\|_{bmo_c} \;=\; \sup_t \|\, T_t|x|^2 - |\, T_t x|^2\|^{1/2} \,.$$

Remark: Ask Tao on BMO-norms. This result is based on work with Perrin on continuous filtration and H_p norms.

The commutative miracle

Theorem: Let $f : \mathbb{R}_+ \to \mathbb{R}$ and $n \in \mathbb{N} \cup \{\infty\}$. TFAE

i) The radial multiplier $M_{f\circ|\,|} : L_\infty(\mathbb{R}^n) \to bmo_c$ is continuous.

ii) For every discrete group G with cocycle of dimension $\dim(H) = n$ the function $m(g) = f(\|b(g)\|)$ is an $LG \to bmo_c$ multiplier.

Proof: By the 'little Grothendieck inequality' the multiplier $M_{f\circ|\,|}$ is completely bounded for the row and column bmo norm. Then we use the Key Formula. ■

Remark: $1_{[0,1]}$ is not a multiplier (Feffermann, Bozejko-Fendler).

Remark: In infinite dimension not many *bmo* multipliers are known, and no good Calderon-Zygmund theory seems to be known.

Remark: By interpolation, we also have L_p boundedness.

Application towards group operators

Theorem:(J.-M.-P.) Let $k : \mathbb{R}^n \to \mathbb{C}$ satisfying $\|\hat{k}\| \leq C_1$ and the Hörmander condition

$$\sup_x \int_{|x-y| \geq 2} |k(y-x) - k(x)| dy \leq C .$$

Let $\lambda(x)(f)(y) = f(x+y)$ the left regular representation. Then the operator

$$\Phi_k(a) = \int_{\mathbb{R}^n} k(x)\lambda(x)a\lambda(-x)dx$$

is bounded on the Schatten class S_p for $1 < p < \infty$.

Problem: Similar extension for quantum groups, which conditions are required? The proof uses $S_p(L_2(\mathbb{R}^n)) = L_p(L_\infty(\mathbb{R}^n) \rtimes \mathbb{R}^n)$ and a beefed up heat semigroup.

Noncommutative version of Mihlin's multiplier result

Theorem (JMP)

Let $n < \infty$ and $\Phi : \mathbb{R}^n \to \mathbb{C}$ be a functions such that

$$|D^\alpha \Phi| \;\le\; \min\{\|\xi\|^{-|\alpha|+\varepsilon}, \|\xi\|^{-|\alpha|-\varepsilon}\}$$

for all $|\alpha| \;\le\; n+2$ and some $\varepsilon > 0$ and

$$\varphi(g) \;=\; \Phi(b(g)) \;.$$

Then m_φ is bounded on L_p for all $1 < p < \infty$.

Remark: If either $\alpha(G)$ is a finite subgroup of $O(n)$ or $b(G)$ is a lattice in $\mathbb{R}^n$, then the ε can be removed and $|\alpha| \;\le\; \lfloor \frac{n}{2} \rfloor + 1$ derivatives are enough. As of now ε is a prize for noncommutativity.
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Sketch of proof

✎ We first consider operators of the form

$$\Phi(f\lambda(g)) \;=\; T_g(f)\lambda(g)\,.$$

✎ For $T_g = T$ a Fourier multiplier, we can prove $L_\infty \to bmo_c$ estimates.

✎ Under additional Schur multiplier assumptions we can also handle variable T_g operators. These occur naturally when trying to estimate the bmo-row norm on the crossed product.

✎ Then we have to cheat: i.e.

$$\|x\|_p \;\le\; c\|A^{-\varepsilon}x\|_{H_p^c}^{1/2}\|A^{\varepsilon}x\|_{H_p^c}^{1/2}$$

for suitable intrinsic semigroup H_p^c spaces.

Problems related Markov dilations and derivations

- The Markov dilation is not unique;
- Indeed, G be a discrete group and H the real space associated with f.d. a cocycle. Let $B : H \otimes L_2(0,\infty) \to \bigcap_{p<\infty} L_p(\Omega)$ be a map which maps vectors to gaussian variables. Then $\pi_t : LG \to L_\infty(\Omega) \rtimes G$ (measure preserving action) given by

$$\pi_t(\lambda(g)) \;=\; e^{iB(h\otimes 1_{[0,t]})}\lambda(g)$$

  gives a Markov dilation.
- For $t = 1$ and $H = \mathbb{R}^n$ we find $\pi : LG \to L_\infty(\mathbb{R}^n, \gamma_n) \rtimes G$ with the gaussian measure.
- In the key formula we have to replace γ_n by the Lebesque measure or the Haar measure of $\widehat{\mathbb{R}^d_n}$ in order to use the ordinary Laplacian. What corresponds to this change of measure for arbitrary semigroups? Is this a flat phenomenon?

Problems related Markov dilations and derivations ...

✏ Alternatively we may work with $\Gamma_0(H \otimes L_2(0,\infty))$, Voiculescu's algebra of semi-circulars. Then there exists a twisted action of G on Γ_0 such that the free Markov dilation is realized by a family of unitaries

$$dw_t(g) \;=\; -\varphi(g)w_t(g)dt+idB(1_{[0,t]}]\otimes b(g))w_t(g) \quad w_0(g) \;=\; 1\;.$$

Note that $dw_t(g)$ is bounded in the free case, and generates a nice "free tangent space". Then $\pi_t^{free}(g) = w_t(g)\lambda(g)$ gives the free Markov dilation.

Problems related Markov dilations and derivations ..., ...

- Let $\pi_t : N \to \mathcal{M}$ be a Markov dilation, $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$, and $\lim_n t_n = 0$. Then
$$\delta(x) \ = \ (\frac{\pi_{t_n}(x) - x}{\sqrt{t_n}})^{\bullet}$$
provides a derivation such that
$$2\Gamma(x,y) \ = \ E_N(\delta(x)^* \delta(y)) \ .$$
- In the key formula and for the gaussian Markov dilation we find the same (nicer) derivation $\delta : LG \to C(\mathbb{R}^n) \rtimes G$ given by
$$\delta(\lambda(g)) \ = \ i(b(g), \cdot)\lambda(g) \ .$$
- There are semigroups on commutative spaces which violate the condition $\Gamma(x,x) \in L_1$ for all $x \in \mathrm{dom}(A^{1/2})$, and hence then $\Gamma(x,x)$ can only be defined in the sense of Sauvageot, Cipriani, Lindsay, Davies with values in $\mathrm{dom}(A^{1/2})^*$ or with the help iof ultraproducts above.

- ✏ In these examples the boundedness of the Riesz transform fails dramatically ($\sqrt{n}$ for semigroup on M_n).
- ✏ In the key formula and for the gaussian Markov dilation we find the same derivation $\delta : LG \to C(\mathbb{R}^n) \rtimes G$ given by

$$\delta(\lambda(g)) \;=\; i(b(g),\cdot)\lambda(g)\;.$$

- ✏ There should be multiplier theorems using higher order derivatives.

Thanks for listening