Report

On the operator $u + \lambda v$ and C^*-subalgebras of the universal irrational rotation algebra

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Major: Mathematics

Operator theory and operator algebras

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OUTLINE

1. SPECTRUM AND BROWN SPECTRUM OF $u + \lambda v$

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2. GENERALIZED UNIVERSAL IRRATIONAL ROTATION C^*-ALGEBRAS

Spectrum

Let θ be an irrational number and let u and v be two unitary operators such that

$$vu = e^{2\pi i\theta} uv. \quad (*)$$

Consider the collection of all irreducible pairs of unitary operators (u_α, v_α) satisfying $(*)$ and form the operators

$$\tilde{u} = \sum \oplus u_\alpha, \quad \text{and} \quad \tilde{v} = \sum \oplus v_\alpha.$$

Let $\mathcal{A}_\theta = C^*(\tilde{u}, \tilde{v})$. Then $\mathcal{A}_\theta$ is called the universal irrational rotational algebra. If $\mathcal{A} = C^*(u, v)$ is any other C^*-algebra satisfying $(*)$, then there is an isomorphism of $\mathcal{A}_\theta$ onto $\mathcal{A}$ which carries $\widetilde{u}$ to u and $\widetilde{v}$ to v.

Spectrum

In mathematical physics, the almost mathieu operator arises in the study of the quantum Hall effect. It is given by

$$(H_{\lambda,\theta,\beta}u)(n) = u(n+1) + u(n-1) + 2\lambda\cos(2\pi(n\theta+\beta))u(n)$$

acting as a self-adjoint operator on the Hilbert space $l^2(\mathbb{Z})$. Here $\theta, \beta, \lambda \in \mathbb{R}$ are parameters.

Almost mathieu operator was firstly introduced by R.Peierls [Pei]. In pure mathematics, its importance come from the fact of being one of the best-understood examples of an ergodic Schrödinger operator. For example, three problems (now all solved) of Barry Simon's fifteen problems [Ba-SI] about Schrödinger operators" for the twenty-first century "featured the almost Mathieu operator.

Spectrum

The fourth problem in [Kac-Sim] (known as the "Ten martini problem" after Kac and Simon) conjectures that the spectrum of the almost Mathieu operator is a cantor set for all $\lambda \neq 0$ and irrational number θ.

Recall that $\mathcal{A}_\theta$ can be represented on $l^2(\mathbb{Z})$, by mapping u into the bilateral shift (taking ϕ into $(\phi(n-1))_{n\in\mathbb{Z}})$, and then v into the operation of multiplication by $e^{2\pi in\theta}$ (taking ϕ into $e^{2\pi in\theta}(\phi(n))_{n\in\mathbb{Z}})$, and then the polynomial $(u+\lambda e^{2\pi i\beta}v)+(u+\lambda e^{2\pi i\beta}v)^*$ is mapped into the bounded self-adjoint operator $H_{\lambda,\theta,\beta}$. Since $\mathcal{A}_\theta$ is simple (where θ is irrational), the spectrum of $H_{\lambda,\theta,\beta}$ is the same as the spectrum of the element :

$$(u+\lambda e^{2\pi i\beta}v)+(u+\lambda e^{2\pi i\beta}v)^*.$$

Spectrum

Thus, the conjecture of Kac and Simon can be reduced to the following:

Conjecture

$$\text{If } a = u + \lambda e^{2\pi i\beta} v + u^* + \overline{\lambda} e^{-2\pi i\beta} v^*, \text{ then } \sigma(a) = \text{Cantor set},$$

where $\lambda \neq 0, vu = e^{2\pi i\theta} uv$, θ-irrational number.

Recently, Avila and Jitomirskaya affirmatively answered this conjecture (2009).

Spectrum

Question

A natural question: What is the spectrum of $u+\lambda e^{2\pi i\beta}v$?

If θ is an irrational number, then by the uniqueness of $\mathcal{A}_\theta$ the spectrum of $u+\lambda e^{2\pi i\beta}v$ is the same as $u+|\lambda|v$.

So from now on, we always assume that $\lambda>0$ and $\beta=0$.

It is well known that there is a unique trace τ on $\mathcal{A}_\theta$. By the GNS-construction, we obtain a faithful representation π of $\mathcal{A}_\theta$ on $L^2(\mathcal{A}_\theta,\tau)$. The weak operator closure of $\pi(\mathcal{A}_\theta)$ is the hyperfinite II_1 factor R. Note that the spectrum of $u+\lambda v$ is same as the spectrum of $\pi(u+\lambda v)$ in R.

Spectrum

Theorem 1

The spectrum of $u + \lambda v$ is given by:
(1) $\sigma(u+v) = \overline{B(0,1)}$;
(2) $\sigma(u+\lambda v) = S^1, \lambda \in (0,1)$;
(3) $\sigma(u+\lambda v) = \lambda S^1, \lambda > 1$.

SPECTRUM

Strongly irreducible operator

An operator T in $\mathscr{L}(\mathscr{H})$ is said to be strongly irreducible if there exists no non-trivial idempotent in $\{T\}^{'}$.

Strongly irreducible operators are generalization of Jordan blocks in matrix algebras.

SPECTRUM

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Relative strongly irreducible operator

Let M be a type II_1 factor, an operator $T \in M$ is said to be a strongly irreducible operator relative to M, if there exists no non-trivial idempotent in $\{T\}^{'} \cap M$.

Spectrum

Lemma

For every irrational number $\theta \in (0,1)$, there exists no nontrivial idempotents in $\{u+v\}' \cap R$, i.e. $u+v$ is relative strongly irreducible in R.

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For every irrational number $\theta \in (0,1)$, there exists no nontrivial idempotents in $\{u+v\}' \cap R$, i.e. $u+v$ is relative strongly irreducible in R.

Corollary

$\sigma(u+v)$ is connected.

SPECTRUM

THEOREM 2

For θ in a second category subset of $[0,1]$, we have $u+v^k$ is strongly irreducible relative to R for all $k=1,2,\ldots$.

Recall that operator T in a type II_1 factor M is called irreducible if $\{T,T^*\}'\cap M=\mathbb{C}I$. i.e., the von-Neumann subalgebra generated by T is an irreducible subfactor of M.

SPECTRUM

By a result of Popa. S. [Pop], every separable type II_1 factor contains an irreducible operator. By definitions, if T is strongly irreducible relative to M, then T is irreducible relative to M.

An operator T is strongly irreducible relative to a type II_1 factor if and only if XTX^{-1} is an irreducible operator relative to M for every bounded invertible operator $X \in M$. However, if T is irreducible relative to M, this is not true in general. The following result shows that an irreducible operator relative to M can be similar to a unitary operator.

SPECTRUM

THEOREM 3

Let θ be an irrational number in $[0,1]$ and let n be any positive integer. Then in the hyperfinite type II_1 factor R there exists a bounded invertible operator x such that

$$W^*(xux^{-1}) = W^*(u+v^n) = W^*(u,v^n)$$

where $uv = e^{2\pi i\theta} vu$.

Let n be a positive integer, we may show $N = W^*(u+v^n) = W^*(u,v^n)$ is an irreducible subfactor of $W^*(u+v) = R$, with Jones index: $[R:N] = n$.

To see this, we notice that

$$R = N \oplus Nv \oplus Nv^2 \oplus \ldots \oplus Nv^{n-1}.$$

SPECTRUM

By Pimsner and S. Popa 's result [Pim-Pop], we have $[R:N]=n$.

On the other hand, by Theorem 3, for θ in a second category subset of $[0,1]$, $u+v^n$ is strongly irreducible relative to R. So for every bounded invertible operator $x\in R$, $x(u+v^n)x^{-1}$ generates an irreducible subfactor $W^*(x(u+v^n)x^{-1}$ of R. What is $[R:W^*(x(u+v^n)x^{-1}]$?

Spectrum

Let $r(u+\lambda v)$ be the spectral radius of $u+\lambda v$. Then

$$r(u+\lambda v) = \lim_{n\to+\infty} ||(u+\lambda v)^n||^{\frac{1}{n}}$$

$$= \lim_{n\to+\infty} ||u^n(1+\lambda\omega)(1+\alpha\lambda\omega)\cdots(1+\alpha^{(n-1)}\lambda\omega)||^{\frac{1}{n}},$$

where $\omega = u^*v$ is a Haar unitary operator and $\alpha = e^{2\pi i\theta}$. Hence,

$$\begin{aligned} ||(u+\lambda v)^n||^{\frac{1}{n}} &= ||(1+\lambda\omega)(1+\alpha\lambda\omega)\cdots(1+\alpha^{(n-1)}\lambda\omega)||^{\frac{1}{n}} \\ &= ||(1+\lambda M_z)(1+\alpha\lambda M_z)\cdots(1+\alpha^{(n-1)}\lambda M_z)||^{\frac{1}{n}} \\ &= \left(\max_{z\in S^1} |(1+\lambda z)(1+\alpha\lambda z)\cdots(1+\alpha^{(n-1)}\lambda z)|\right)^{\frac{1}{n}} \end{aligned}$$

Spectrum

Lemma

Suppose $0 < \lambda \leq 1$. For any $\varepsilon > 0$, there exists $x \in [0,1]$ and $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left(\prod_{k=0}^{n-1}(1+\lambda^2+2\lambda\cos(2\pi(x+k\theta)))\right)^{\frac{1}{2n}} \geqslant 1-\varepsilon.$$

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Proof

Key point: Let $T : x \to x + \theta(\mathrm{mod} 1)$. Then T is a measure preserving ergodic transformation of $[0,1]$. By Birkhoff's Ergodic theorem, for almost all $x \in [0,1]$, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\ln(1+\lambda^2+2\lambda\cos(2\pi(x+k\theta)) = \int_0^1 \ln(1+\lambda^2+2\lambda\cos 2\pi x)dx = 0.$$

Spectrum

Corollary

$r(u+\lambda v) \geqslant 1, \forall 0 < \lambda \leq 1.$

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Lemma

Suppose $0 < \lambda \leq 1$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in [0,1]$

$$\left(\prod_{k=0}^{n-1}(1+\lambda^2+2\lambda\cos(2\pi(x+k\theta)))\right)^{\frac{1}{2n}} \leq e^{2\varepsilon}, \forall n \geq N.$$

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Corollary

$r(u+\lambda v) \geqslant 1, \forall 0 < \lambda \leq 1.$

Lemma

Suppose $0 < \lambda \leq 1$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in [0,1]$

$$\left(\prod_{k=0}^{n-1}(1+\lambda^2+2\lambda\cos(2\pi(x+k\theta)))\right)^{\frac{1}{2n}} \leq e^{2\varepsilon}, \forall n \geq N.$$

Corollary

$r(u+\lambda v) = 1, \forall 0 < \lambda \leq 1.$

Spectrum

Notice that, $u + v = u(1 + u^* v)$. Since $u^* v$ is a Haar unitary operator, $-1 \in \sigma(u^* v)$. This implies that $u + v$ is not invertible and therefore $0 \in \sigma(u + v)$. Observe that $\sigma(u + v)$ is rotation symmetric. Since $\sigma(u + v)$ is connected and $r(u + v) = 1$, $\sigma(u + v) = \overline{B(0, 1)}$.

For $0 < \lambda < 1$, it is obvious that $0 \notin \sigma(u + \lambda v)$. We can also show that $r((u + \lambda v)^{-1}) = 1$. Then

$$\sigma(u + \lambda v) = S^1.$$

For $\lambda > 1$, consider $\lambda(\frac{1}{\lambda} u + v)$, we have $\sigma(u + \lambda v) = \lambda S^1$.

Brown's Spectrum distribution

Let M be a finite von-Neumann algebra with a faithful normal tracial state τ. The Fuglede-Kadison determinant [Fu-Ka] $\triangle : M \to [0, \infty]$ is given by $\triangle(T) = \exp\{\tau(\ln|T|)\}$, $T \in M$, with $\exp\{-\infty\} := 0$.

For an arbitrary element T in M, the function $\lambda \to \ln(\triangle(T - \lambda I))$ is subharmonic on $\mathbb{C}$, and its Laplacian:

$$d\mu_T(\lambda) = \frac{1}{2\pi}\nabla^2 \ln \triangle(T - \lambda I)$$

In the distribution sense, defines a probability measure μ_T on $\mathbb{C}$, called the Brown's spectral distribution or Brown's measure of T.

Brown's Spectrum distribution

From the definition, Brown measure μ_T only depends on the joint distribution of T and T^*.

If T is normal, then μ_T is the trace τ composed with the spectral projection of T. If $M = M_n(\mathbb{C})$, then μ_T is the normalization counting measure $\frac{1}{n}(\delta_{\lambda_1} + \delta_{\lambda_2} + \ldots + \delta_{\lambda_n})$, where $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of T repeated according to root multiplicity.

Haagerup-Schultz Theorem

Let $T \in M$. If the support set of Brown's measure of T contains more than one point, then T has a nontrivial invariant subspace relative to M.

By this theorem, we can see that it is important to calculate the Brown spectrum of operators.

Brown's Spectrum distribution

Theorem 4

If $0 < \lambda \leq 1$, then the Brown measure of $u + \lambda v$ is the Haar measure on the unit circle S^1. If $\lambda > 1$, then the Brown measure of $u + \lambda v$ is the Haar measure on λS^1.

Brown's Spectrum distribution

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Sketch proof 1

We need the following lemma first,
Lemma (Haagerup-Schultz) Let $T \in M$ and for any $n \in \mathbb{N}$, let $\mu_n \in Prob([0, \infty])$ denote the distribution of $(T^n)^* T^n$ w.r.t τ, and let ν_n denote the push forward measure of μ_n under the map $t \to t^{\frac{1}{n}}$. Moreover, let ν denote the push forward measure of μ_T under the map $z \to |z|^2$, i.e., ν is determined by $\nu([0, t^2]) = \mu_T(\overline{B(0, t)}), t > 0$. Then $\nu_n \to v$ weakly in $Prob([0, \infty))$.

Brown's Spectrum distribution

Sketch proof 2

Let $T = u + \lambda v$, and let ν and ν_n be the measures defined as in Lemma of Haagerup-Schultz. Note that $((T^n)^* T^n)^{\frac{1}{n}} = |(1+\omega)\ldots(1+\alpha^{n-1}\omega)|^{\frac{2}{n}}$, where $\omega = u^* v$ is a Haar unitary operator. So we can view $((T^n)^* T^n)^{\frac{1}{n}}$ as the multiplication operator on $L^2[0,1]$ corresponding to the function

$$|\prod_{k=0}^{n-1}(2 + 2\cos(2\pi(x + k\theta)))|^{\frac{1}{n}}.$$

Let m be the Lebesgue measure on $[0,1]$. For $0 < b < 1$, since $[0,b)$ is an open set relative to $[0,\infty)$ and $\nu_n \to \nu$ weakly in $Prob([0,\infty))$,

Brown's Spectrum distribution

$$\begin{aligned}\nu([0,b)) &\leq \lim_{n\to\infty}\inf \nu_n([0,b)) \\ &= \lim_{n\to\infty}\inf m\left(\left\{x : |\prod_{k=0}^{n-1}(2+2\cos(2\pi(x+k\theta)))|^{\frac{1}{n}} \in [0,b)\right\}\right).\end{aligned}$$

Note that for almost all $x \in [0,1]$,

$$\lim_{n\to\infty} |\prod_{k=0}^{n-1}(2+2\cos(2\pi(x+k\theta)))|^{\frac{1}{n}} = 1.$$

Brown's Spectrum distribution

In particular, $|\prod_{k=0}^{n-1}(2+2cos(2\pi(x+k\theta)))|^{\frac{1}{n}}|$ converges in measure to the constant function 1 on $[0,1]$. Since $b<1$, $\nu([0,b))=0$. Thus ν is the Dirac measure δ_1 and the support of μ_T is contained in S^1. Since μ_T is rotation invariant, μ_T is the Haar measure on S^1.

For $\lambda \neq 1$, by Theorem 1, $\sigma(u+\lambda v)=S^1$ if $0<\lambda<1$ and $\sigma(u+\lambda v)=\lambda S^1$ if $\lambda>1$. Since $\mu_{(u+\lambda v)}$ is rotation invariant and the support of $\mu_{(u+\lambda v)}$ is contained in $\sigma(u+\lambda v)$, the Brown measure of $u+\lambda v$ is the Haar measure on the unit circle S^1 if $0<\lambda<1$ and the Haar measure on λS^1 if $\lambda>1$.

Generalized universal irrational rotation C^*-algebras

To study $C^*(u+\lambda v)$ we introduce the following definition. A generalized universal irrational rotation algebra $\mathcal{A}_{\theta,\gamma} = C^*(x,w)$ is the universal C^*-algebra satisfying the following properties:

$$w^*w = ww^* = 1, \tag{1}$$

$$x^*x = \gamma(w), \tag{2}$$

$$xx^* = \gamma(e^{-2\pi i\theta}w), \tag{3}$$

$$xw = e^{-2\pi i\theta}wx, \tag{4}$$

where $\gamma(z) \in C(S^1)$ is a positive function.

TRACIAL LINEAR FUNCTIONALS ON GENERALIZED UNIVERSAL IRRATIONAL ROTATION C^*-ALGEBRAS

- There is a classical faithful trace τ on a generalized universal irrational algebra $\mathcal{A}_{\theta,\gamma}$

Tracial linear functionals on generalized universal irrational rotation C^*-algebras

- There is a classical faithful trace τ on a generalized universal irrational algebra $\mathcal{A}_{\theta,\gamma}$
- Applying the GNS-construction to τ, $\mathcal{A}_{\theta,\gamma} = C^*(x, w)$ is isomorphic to $C^*(u\gamma(v)^{1/2}, v)$ and the isomorphism takes x to $u\gamma(v)^{1/2}$ and w to v.

Tracial linear functionals on generalized universal irrational rotation C^*-algebras

- There is a classical faithful trace τ on a generalized universal irrational algebra $\mathcal{A}_{\theta,\gamma}$
- Applying the GNS-construction to τ, $\mathcal{A}_{\theta,\gamma} = C^*(x, w)$ is isomorphic to $C^*(u\gamma(v)^{1/2}, v)$ and the isomorphism takes x to $u\gamma(v)^{1/2}$ and w to v.
- Suppose $\gamma(z)$ has finite zero points which can be divided into nonempty disjoint classes $A_1, \cdots, A_r$ in the following sense: z_1 and z_2 in A_j if and only if $z_2 = e^{2\pi ik\theta}z_1$ for some $k \in \mathbb{Z}$. Then the dimension of the space of tracial linear functionals on $\mathcal{A}_{\theta,\gamma} = C^*(x, w)$ is $1 + \sum_{j=1}^{r}(|A_j| - 1)$, where $|A_j|$ is the number of elements in A_j.

Simple generalized universal irrational rotation C^*-algebras

- Let Λ_1 be the set of zero points of functions $\gamma(e^{2\pi in\theta}z)$ for $n \geq 0$, and let Λ_2 be the set of zero points of functions $\gamma(e^{-2\pi in\theta}z)$ for $n \geq 1$. Then $\mathcal{A}_{\theta,\gamma}$ is a simple algebra if and only if $\Lambda_1 \cap \Lambda_2 = \emptyset$.

Simple generalized universal irrational rotation C^*-algebras

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- If a generalized universal C^*-algebra $\mathcal{A}_{\theta,\gamma}$ is simple, then for every α in $(\mathbb{Z} + \mathbb{Z}\theta) \cap [0, 1]$, there is a projection p in $\mathcal{A}_{\theta,\gamma}$ such that $\tau(p) = \alpha$.

Simple generalized universal irrational rotation C^*-algebras

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- If a generalized universal C^*-algebra $\mathcal{A}_{\theta,\gamma}$ is simple, then for every α in $(\mathbb{Z} + \mathbb{Z}\theta) \cap [0, 1]$, there is a projection p in $\mathcal{A}_{\theta,\gamma}$ such that $\tau(p) = \alpha$.
- Suppose θ and η are two irrational numbers and $\gamma \in C(S^1)$ is a positive function with finite zero points. Then $\mathcal{A}_{\theta,\gamma} \cong \mathcal{A}_{\eta,\gamma}$ if and only if $\theta \equiv \pm\eta \mod \mathbb{Z}$.

K-groups of generalized universal irrational rotation C^*-algebras

Theorem 5

Suppose $\gamma(z)$ has n zero points, $n \geq 1$. Then

$$K_0(\mathcal{A}_{\theta,\gamma}) \cong \mathbb{Z}^{n+1}, \quad K_1(\mathcal{A}_{\theta,\gamma}) \cong \mathbb{Z}.$$

Sketch of proof

K-groups of generalized universal irrational rotation C^*-algebras

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Sketch of proof

- Suppose Λ is the set of zero points of $\gamma(z)$. Let A be the C^*-subalgebra generated by w, let

$$J_1 = \{f(w) : \ f(\lambda) = 0 \text{ for } \lambda \in \Lambda\},$$

  and let $J_2 = uJ_1u^*$. Then $\mathcal{A}_{\theta,\gamma}$ is $*$-isomorphic to the covariance algebra $C^*(A, \Theta)$ for the partial automorphism $\Theta = (\mathrm{Ad}u, J_1, uJ_1u^*)$ of $C^*(w)$ in the sense of Ruy Exel [9].

K-groups of generalized universal irrational rotation C^*-algebras

- For a covariance algebra $C^*(A, \Theta)$ for the partial automorphism $\Theta = (\theta, J, \theta(J))$ of A, Ruy Exel [9] proved the following generalized Pimsner-Voiculescu exact sequnece

$$
\begin{array}{ccccc}
K_0(J) & \xrightarrow{i_* - \theta_*^{-1}} & K_0(A) & \xrightarrow{i_*} & K_0(\mathcal{A}_{\theta,\gamma}) \\
\uparrow & & & & \downarrow \\
K_1(\mathcal{A}_{\theta,\gamma}) & \xleftarrow{i_*} & K_1(A) & \xleftarrow{i_* - \theta_*^{-1}} & K_1(J)
\end{array}
$$

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Corollary

Suppose $\gamma_1(z)$ and $\gamma_2(z)$ have finite zero points. Then $\mathcal{A}_{\theta,\gamma_1} \cong \mathcal{A}_{\theta,\gamma_1}$ implies that $\gamma_1(z)$ and $\gamma_2(z)$ have same number of zero points

Universal irrational rotation C^*-algebras

It is well known that the universal irrational C^*-algebra $\mathcal{A}_\theta$ has the following properties (Rieffel [1981], Pimsner-Voiculescu [1980], Elliott-Evans [1993])

(ȧ) $\mathcal{A}_\theta$ is simple.

(ḃ) There is a unique trace on $\mathcal{A}_\theta$.

(ċ) For every α in $(\mathbb{Z}+\mathbb{Z}\theta)\cap[0,1]$, there exists projection p in $C^*(u,v)$ such that $\tau(p)=\alpha$.

(ḋ) $K_0(\mathcal{A}_\theta)\cong\mathbb{Z}+\mathbb{Z}\theta$ and $K_1(\mathcal{A}_\theta)\cong\mathbb{Z}^2$.

(ḟ) $\mathcal{A}_\theta\cong\mathcal{A}_\eta\Leftrightarrow\theta=\pm\eta$ (mod $\mathbb{Z}$).

(ġ) $\mathcal{A}_\theta$ is a limit circle algebra and hence with stable rank 1 and real rank 0.

GENERALIZED UNIVERSAL IRRATIONAL ROTATION C^*-ALGEBRAS

Now, we consider C^*-algebra $C^*(u+\lambda v)$ generated by $u+\lambda v$. We have the following:

(a) For $\lambda \neq 1$, $0 < \lambda < +\infty$, we have $C^*(u+\lambda v) = C^*(u,v)$. However, $C^*(u+v)$ is a proper subalgebra of $\mathcal{A}_\theta = C^*(u,v)$. Indeed, $\inf\{||u-x|| : x \in C^*(u+v)\} = 1$.

(b) $C^*(u+v) \cong \mathcal{A}_{\theta,\gamma}$ for $\gamma(z) = |1+z|^2$.

(c) $C^*(u+v)$ is simple.

(d) For every $\alpha \in (\mathbb{Z}+\mathbb{Z}\theta) \cap [0,1]$, there exists projection p in $C^*(u+v)$ such that $\tau(p) = \alpha$.

(e) $K_0(C^*(u_\theta + v_\theta)) \cong \mathbb{Z}+\mathbb{Z}\theta$ and $K_1(C^*(u+v)) \cong \mathbb{Z}$. In particular, $C^*(u+v)$ is not $*$-isomorphic to $C^*(u,v)$.

(f) $C^*(u_\theta + v_\theta) \cong C^*(u_\eta + v_\eta) \Leftrightarrow \theta = \pm\eta (\text{mod}\mathbb{Z})$.

Future plan

Question 1: Is $C^*(u+v)$ a limit circle C^*-algebra?
Question 2: Classify generalized universal irrational rotation algebras $\mathcal{A}_{\theta,\gamma}$, presumably in terms of zero points of $\gamma(z)$.

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