Calderón-Zygmund operators associated to matrix-valued kernels

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Outline :

1. Motivation
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- A *semicommutative* CZO has the formal expression

$$Tf(x) \sim \int_{\mathbb{R}^n} k(x,y)(f(y))\, dy,$$

where the kernel acts linearly on the matrix-valued function $f = (f_{ij})$ and satisfies standard size/smoothness Calderón-Zygmund type conditions.

- This is the operator model for quite a number of problems which have attracted some attention in recent years, including matrix-valued paraproducts , operator-valued Calderón-Zygmund theory or Fourier multipliers on group von Neumann algebras .
- To be more precise, we consider noncommutative L_p spaces on the von Neumann algebra $\mathcal{A}$ formed by essentially bounded functions $f : \mathbb{R}^n \to \mathcal{B}(\ell_2)$. Let us highlight a few significant examples :

- **Scalar kernels.** $k(x,y) \in \mathbb{C}$ and

$$k(x,y)(f(y)) \; = \; \Big(k(x,y) f_{ij}(y) \Big).$$

- **Schur product actions.** $k(x,y) \in \mathcal{B}(\ell_2)$ and

$$k(x,y)(f(y)) \; = \; \Big(k_{ij}(x,y) f_{ij}(y) \Big).$$

- **Fully noncommutative model.** $k(x,y) \in \mathcal{B}(\ell_2) \bar{\otimes} \mathcal{B}(\ell_2)$ and

$$k(x,y)(f(y)) \; = \; \Big(\sum\nolimits_m \mathrm{tr} \big(k''_m(y) f(y) \big) k'_m(x)_{ij} \Big).$$

- **Partial traces, noncommuting kernels.** $k(x,y) \in \mathcal{B}(\ell_2)$ and

$$k(x,y)(f(y)) \; = \; \left\{ \begin{array}{l} \Big(\sum_s k_{is}(x,y) f_{sj}(y) \Big), \\ \Big(\sum_s f_{is}(y) k_{sj}(x,y) \Big). \end{array} \right.$$

- Scalar kernels required in Parcet-2009 a matrix-valued Calderón-Zygmund decomposition in terms of noncommutative martingales and a pseudo-localization principle to control the tails of Tf in the L_2-metric. Hilbert space valued kernels were later considered in Mei-Parcet-2009, see also Pisier-Xu-1997, Mei-2007, Radrianantoanina-2002,2005,2007 for previous related results.
- The second case refers to the Schur matrix product $k(x,y) \bullet f(y)$, considered for the first time in Junge-Mei-Parcet-arxiv to analyze cross product extensions of classical CZO's. It is instrumental for Hörmander-Mihlin type theorems on Fourier multipliers associated to discrete groups and for Schur multipliers with a Calderón-Zygmund behavior, see also Junge-Mei-Parcet-preprint1, preprint2.

- In the fully noncommutative model, we approximate $k(x,y)$ by a sum of elementary tensors $\sum_m k'_m(x) \otimes k''_m(y)$ and the action is given by

$$Tf(x) \ \sim \ \int_{\mathbb{R}^n} (id \otimes \mathrm{tr}) \Big[k(x,y) \big(\mathbf{1} \otimes f(y) \big) \Big] \, dy.$$

  In this case, the noncommutative nature of $L_p(\mathcal{A})$ predominates and the presence of a Euclidean subspace is ignored. That is what happens for purely noncommutative CZO's in Junge-Mei-Parcet-preprint2.

- The last case refers to matrix-valued kernels acting on f by left/right multiplication, $k(x,y)f(y)$ and $f(y)k(x,y)$. Matrix-valued paraproducts are prominent examples. This is the only case in which the kernel does not commute with f.

- Our main goal is to obtain endpoint estimates for CZO's with noncommuting kernels, motivated by a recent estimate from Junge-Mei-Parcet-arxiv for semicommutative CZO's. If $k(x,y)$ acts linearly on $\mathcal{B}(\ell_2)$ and satisfies the Hörmander smoothness condition in the norm of bounded linear maps on $\mathcal{B}(\ell_2)$, Lemma 1.3 in that paper can be summarized as follows
  - If T is $L_\infty(L_2^r(\mathbb{R}^n)$-bounded, then $T : L_\infty(\mathcal{A}) \to \mathrm{BMO}_r(\mathcal{A})$,
  - If T is $L_\infty(L_2^c(\mathbb{R}^n)$-bounded, then $T : L_\infty(\mathcal{A}) \to \mathrm{BMO}_c(\mathcal{A})$.
- Here, the $L_\infty(L_2^c)$-boundedness assumption refers to

$$\Big\| \Big(\int_{\mathbb{R}^n} Tf(x)^* Tf(x)\, dx \Big)^{\frac12} \Big\|_{\mathcal{B}(\ell_2)} \lesssim \Big\| \Big(\int_{\mathbb{R}^n} f(x)^* f(x)\, dx \Big)^{\frac12} \Big\|_{\mathcal{B}(\ell_2)},$$

while the column-BMO norm of a matrix-valued function g is given by

$$\sup_{Q \text{ cube}} \Big\| \Big(\fint_Q \big(g(x) - g_Q\big)^* \big(g(x) - g_Q\big)\, dx \Big)^{\frac12} \Big\|_{\mathcal{B}(\ell_2)}.$$

- Taking adjoints —so that the $*$ switches everywhere from left to right— we find $L_\infty(L_2^r)$-boundedness and the row-BMO norm. The noncommutative BMO space $\mathrm{BMO}(\mathcal{A}) = \mathrm{BMO}_r(\mathcal{A}) \cap \mathrm{BMO}_c(\mathcal{A})$ was introduced in Pisier-Xu-1997. According to Musat-2003 it has the expected interpolation behavior in the L_p scale. Thus, standard interpolation and duality arguments show that $T : L_p(\mathcal{A}) \to L_p(\mathcal{A})$ for $1 < p < \infty$ provided the kernel is smooth enough in both variables and T is a normal self-adjoint map satisfying the $L_\infty(L_2^r)$ and $L_\infty(L_2^c)$ boundedness assumptions. In other words, the row/column boundedness conditions essentially play the role of the L_2-boundedness assumption in classical Calderón-Zygmund theory.

- Although this certainly works for non-scalar kernels, e.g. Schur product actions were used in Theorem B of Junge-Mei-Parcet-arxiv, the boundedness assumptions impose nearly commuting conditions on the kernel which are too strong for CZO's associated to noncommuting kernels.
- Namely, given $k : \mathbb{R}^{2n} \setminus \Delta \to \mathcal{B}(\ell_2)$ smooth and given $x \notin \mathrm{supp}_{\mathbb{R}^n} f$, let us set formally the row/column CZO's

$$T_c f(x) = \int_{\mathbb{R}^n} k(x,y) f(y)\, dy \quad \text{and} \quad T_r f(x) = \int_{\mathbb{R}^n} f(y) k(x,y)\, dy.$$

It is not difficult to construct noncommuting kernels with

i) T_r and T_c are $L_2(\mathcal{A})$-bounded,
ii) T_r and T_c are not $L_p(\mathcal{A})$-bounded for $1 < p \neq 2 < \infty$,

see e.g. Section 6.1 of Parcet-2009 for specific examples.

- Assume for what follows that T_r and T_c are $L_2(\mathcal{A})$-bounded. We are interested in weakened forms of L_p boundedness and endpoint estimates for these CZO's. A *dyadic noncommuting* CZO will be a $L_2(\mathcal{A})$-bounded pair (T_r, T_c) associated to a noncommuting kernel satisfying one of the following conditions :
  a) Perfect dyadic kernels
  $$\big\|k(x,y) - k(z,y)\big\|_{\mathcal{B}(\ell_2)} + \big\|k(y,x) - k(y,z)\big\|_{\mathcal{B}(\ell_2)} = 0$$
  whenever $x, z \in Q$ and $y \in R$ for some disjoint dyadic cubes Q, R.
  b) Cancellative Haar shift operators
  $$k(x,y) = \sum_{Q \text{ dyadic}} \sum_{\substack{R,S \text{ dyadic} \subset Q \\ \ell(R)=2^{-r}\ell(Q) \\ \ell(S)=2^{-s}\ell(S)}} \alpha_{RS}^{Q} h_R(x) h_S(y),$$
  where the $\alpha_{RS}^{Q} \in \mathcal{B}(\ell_2)$ with $\|\alpha_{RS}^{Q}\|_{\mathcal{B}(\ell_2)} \leq \frac{\sqrt{|R||S|}}{|Q|}$. Here h_Q refers to any of the $2^n - 1$ Haar functions related to Q.

- Perfect dyadic kernels were introduced in Auscher-Hoffman-Muscalu-Tao-Thiele-2002 and include Haar multipliers, as well as paraproducts and their adjoints. If J_- and J_+ denote the left/right halves of a dyadic interval in $\mathbb{R}$, the standard model for Haar shifts is the dyadic Hilbert transform with kernel $\sum_J (h_{J_-}(y) - h_{J_+}(y)) h_J(x)$. It appeared after Petermichl's crucial result in 2000, showing the classical Hilbert transform as a certain average of dyadic Hilbert transforms. Hytönen's representation theorem to appear in Ann.Math extends this result to arbitrary CZO's.
- We will write *generic noncommuting* CZO for $L_2(\mathcal{A})$-bounded pairs (T_r, T_c) with a noncommuting kernel satisfying the standard smoothness. Our first significant result is the following.

Theorem A

The following inequalities hold:

i) *Dyadic noncommuting CZO's. Given* $f \in L_1(\mathcal{A})$

$$\inf_{f=f_r+f_c} \big\| T_r f_r \big\|_{1,\infty} + \big\| T_c f_c \big\|_{1,\infty} \lesssim \|f\|_1.$$

ii) *Generic noncommuting CZO's. Given* $f \in \mathrm{H}_1(\mathcal{A})$

$$\inf_{f=f_r+f_c} \big\| T_r f_r \big\|_1 + \big\| T_c f_c \big\|_1 \lesssim \|f\|_{\mathrm{H}_1(\mathcal{A})}.$$

- Our main result is the inequality given in Theorem A i) and their noncommutative generalizations in Theorem C below. The left/right modular nature of T_r/T_c is essential for the weak type $(1,1)$ estimates. The following result easily follows from Theorem A by interpolation/duality. Nevertheless, it is worth mentioning the L_p inequalities that we find.

Theorem B

The following inequalities hold for generic noncommuting CZO's:

i) *If* $1 < p < 2$ *and* $f \in L_p(\mathcal{A})$

$$\inf_{f=f_r+f_c} \big\| T_r f_r \big\|_p + \big\| T_c f_c \big\|_p \lesssim \|f\|_p.$$

ii) *If* $2 < p < \infty$ *and* $f \in L_p(\mathcal{A})$

$$\big\| T_r f \big\|_{\mathrm{H}_p^r(\mathcal{A})} + \big\| T_c f \big\|_{\mathrm{H}_p^c(\mathcal{A})} \lesssim \|f\|_p.$$

iii) *Given* $f \in L_\infty(\mathcal{A})$, *we also have* $\|T_r f\|_{\mathrm{BMO}_r(\mathcal{A})} + \|T_c f\|_{\mathrm{BMO}_c(\mathcal{A})} \lesssim \|f\|_\infty.$

- Theorems A and B also hold for other operator-valued functions, replacing $\mathcal{B}(\ell_2)$ by any semifinite von Neumann algebra $\mathcal{M}$.

- Let us now consider a weak-$*$ dense filtration $\Sigma_{\mathcal{A}} = (\mathcal{A}_n)_{n\geq 1}$ of von Neumann subalgebras of an arbitrary semifinite von Neumann algebra $\mathcal{A}$. In the following result, we will consider two kind of operators in $L_p(\mathcal{A})$:

  a) Noncommuting martingale transforms

$$M_\xi^r f = \sum_{k\geq 1} \Delta_k(f)\xi_{k-1} \quad \text{and} \quad M_\xi^c f = \sum_{k\geq 1} \xi_{k-1}\Delta_k(f).$$

  b) Paraproducts with noncommuting symbol

$$\Pi_\rho^r(f) = \sum_{k\geq 1} \mathsf{E}_{k-1}(f)\Delta_k(\rho) \quad \text{and} \quad \Pi_\rho^c(f) = \sum_{k\geq 1} \Delta_k(\rho)\mathsf{E}_{k-1}(f).$$

- Here Δ_k denotes the martingale difference operator $\mathsf{E}_k - \mathsf{E}_{k-1}$ and $\xi_k \in \mathcal{A}_k$ is an adapted sequence. Of course, the symbols ξ and ρ do not necessarily commute with the function. Randrianantoanina considered noncommutative martingale transforms with commuting coefficients in 2002.
- As for paraproducts with noncommuting symbols, Mei studied the L_p-boundedness for $p > 2$ and regular filtrations in 2006 and also analyzed in 2010 the case $p < 2$ in the dyadic matrix-valued case under a strong BMO condition of the symbol. Our theorem below goes beyond these results, see also Mei-Parcet-2009 for related results.

Theorem C

Consider the pairs:

i) *Martingale transforms* (M_ξ^r, M_ξ^c), *with* $\sup_k \|\xi_k\|_{\mathcal{M}} < \infty$.

ii) *Martingale paraproducts* (Π_ρ^r, Π_ρ^c), *with* $\Pi_\rho^{r/c}$ $L_2(\mathcal{A})$*-bounded.*

If $\Sigma_{\mathcal{A}}$ *is regular, we obtain weak type* $(1,1)$ *inequalities like in Theorem Ai) for martingale transforms and paraproducts . The estimates in Theorems Aii) and B also hold for both families and for arbitrary filtrations* $\Sigma_{\mathcal{A}}$*. Moreover, the martingale paraproducts* Π_ρ^r *and* Π_ρ^c *are* L_p*-bounded for* $2 < p < \infty$ *and* $L_\infty \to \mathrm{BMO}$.

- Our results recover those in Randrianantoanina-2002 and provide appropriate substitutes for noncommuting coefficients. Our result for paraproducts goes beyond Mei-2006's in two aspects. First, our estimates for $p > 2$ hold for arbitrary martingales, not just for regular ones. Second, we give a partial answer to Mei's question for the case $1 \leq p < 2$.

- The idea for the proof of Theorem Ai) is to first decompose f as $f = f_c + f_r$ by triangular truncation, then to write each part f_i as the sum of g_d^i, g_{off}^i, b_d^i, b_{off}^i using the noncommutative Calderón-Zygmund decomposition. Finally, the localization property of perfect dyadic map or Haar shift enable us to deal with the terms g_{off}^i, b_d^i, b_{off}^i, which is not the case for general map.

  Problem 1. Extend Theorem Ai) to arbitrary CZO's with noncommuting kernels.

- As explained in Parcet-2007, a key ingredient for a successful application of the noncommutative CZ decomposition is to use it on $\mathcal{M}$-bimodular maps. In this paper, our decomposition $f = f_r + f_c$ has allowed us to make it work for either left or right $\mathcal{M}$-modular maps. There are however many other semicommutative CZO's, some of which were mentioned in the Motivation. We know from Junge-Mei-Parcet-arxiv that a semicommutative CZO satisfying $L_\infty(L_2^r)$ and $L_\infty(L_2^c)$ boundedness also satisfies $T : L_\infty(\mathcal{A}) \to \mathrm{BMO}(\mathcal{A})$.

  Problem 2. Do we have $T : L_1(\mathcal{A}) \to L_{1,\infty}(\mathcal{A})$ under the same assumptions ?

- It is a little bit unsatisfactory to require regular filtrations to provide weak type inequalities for martingales transforms/paraproducts with noncommuting coefficients/symbols. It is well-known that these estimates hold in the classical setting for any filtration by means of Gundy's decomposition. The noncommutative extension of Gundy's decomposition was constructed in Parcet-Randrianantoanina-2006.

  Problem 3. Can we eliminate the regularity assumption from Theorem Ci) ?

Thank you