$\mathfrak{ad}$-*invariant Lévy processes on free quantum groups and their Markov semigroups*

Uwe Franz (Université de Franche-Comté)

June 4-10, 2012, 武汉大学

joint work with:
Anna Kula (Jagiellonian University Kraków)
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Goal

- Identify nice[1] quantum Markov semigroups (i.e. semigroups of UCP maps) on C*-algebras of compact quantum groups.
- Study their "potential theory", i.e. construct their Dirichlet forms, derivations, Dirac operators, etc., and study their spectral properties.

[1] analoges of Browian motion and the heat semigroup

Outline

- The free orthogonal quantum group
- translation invariant Markov semigroups
- $\mathfrak{ad}$-invariance
- Example: a Lévy-Khinchine-type formula for $\mathfrak{ad}$-invariant convolution semigroups on O_n^+

The orthogonal group O_n

Theorem

The C*-algebra $C(O_n)$ of continuous functions on the orthogonal group O_n is the universal commutative C*-algebra generated by

$$x_{jk} \qquad 1 \le j,k \le n$$

with the relations

$$x_{jk}^* = x_{jk}$$
$$\sum_{\ell=1}^n x_{j\ell}x_{k\ell} = \delta_{jk} = \sum_{\ell=1}^n x_{\ell j}x_{\ell k}$$

The free orthogonal quantum group O_n^+

Definition (Wang)

The (universal or full) C*-algebra $C_u(O_n^+)$ (also denoted $A_o(I_n)$ or $A_o(n)$) of "continuous functions" on the **free orthogonal quantum group** O_n^+ is defined as the universal C*-algebra generated by

$$x_{jk} \qquad 1 \leq j, k \leq n$$

with the relations

$$x_{jk}^* = x_{jk}$$

$$\sum_{\ell=1}^{n} x_{j\ell} x_{k\ell} = \delta_{jk} = \sum_{\ell=1}^{n} x_{\ell j} x_{\ell k}$$

Compact Quantum Groups: definition

Definition (Woronowicz)

A **compact quantum group** is a pair $\mathbb{G} = (\mathsf{A}, \Delta)$, where A is a unital C^*-algebra, $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ is a unital, $*$-homomorphism which is coassociative (i.e. $(\Delta \otimes \mathrm{id}_\mathsf{A}) \circ \Delta = (\mathrm{id}_\mathsf{A} \otimes \Delta) \circ \Delta$) such that the quantum cancellation rules are satisfied

$$\overline{\mathrm{Lin}}((1 \otimes \mathsf{A})\Delta(\mathsf{A})) = \overline{\mathrm{Lin}}((\mathsf{A} \otimes 1)\Delta(\mathsf{A})) = \mathsf{A} \otimes \mathsf{A}.$$

A is called the algebra of "continuous functions" on $\mathbb{G}$ and denoted by $C(\mathbb{G})$.

O_n^+ is a compact quantum group

Remark

There exists a unique unital $*$-algebra homomorphism $\Delta : C_u(O_n^+) \to C_u(O_n^+) \otimes C_u(O_n^+)$ with

$$\Delta(x_{jk}) = \sum_{\ell=1}^{n} x_{j\ell} \otimes x_{\ell k}.$$

$O_n^+ = (C_u(O_n^+), \Delta)$ is a compact quantum group.

The Haar state

For $a \in \mathrm{A} = C(\mathbb{G})$ and $\xi, \xi' \in \mathrm{A}^*$ we can define a **convolution**

$$\begin{aligned} \xi \star \xi'(a) &= (\xi \otimes \xi')\Delta(a) \\ \xi \star a &= (\mathrm{id} \otimes \xi)\Delta(a) \\ a \star \xi &= (\xi \otimes \mathrm{id})\Delta(a) \end{aligned}$$

Theorem (Woronowicz)

Let (A, Δ) be a compact quantum group. There exists unique state (called the **Haar state**) h on A such that

$$a \star h = h \star a = h(a)I, \quad a \in \mathrm{A}.$$

In general, h is not a trace.

- An n-**dimensional unitary corepresentation** of $\mathbb{G}$ is a unitary $U = (u_{jk})_{1\le j,k\le n} \in M_n(\mathrm{A})$ such that

$$\Delta(u_{jk}) = \sum_{\ell=1}^{n} u_{j\ell} \otimes u_{\ell k}.$$

- Let $(U^{(s)})_{s\in\mathcal{I}}$ be a complete family of mutually inequivalent irreducible unitary correpresentations of $\mathbb{G}$. The **algebra of the "polynomials" (or "smooth functions") on** $\mathbb{G}$ is defined as

$$\mathrm{Pol}(\mathbb{G}) = \mathrm{Lin}\{u^{(s)}_{jk}; s \in \mathcal{I}, 1 \le j, k \le n_s\},$$

where n_s is the dimension of $u^{(s)}$.

- $\mathrm{Pol}(\mathbb{G})$ is a dense $*$-subalgebra of $C(\mathbb{G})$, which is a Hopf $*$-algebra with

$$\varepsilon\left(u^{(s)}_{jk}\right) = \delta_{jk} \quad \text{and} \quad S\left(u^{(s)}_{jk}\right) = \left(u^{(s)}_{kj}\right)^{*}.$$

Two more algebras of "functions" on O_n^+

The reduced C^*-algebra $C_r(O_n^+)$ of "cont. functions" on O_n^+

For $n \geq 3$ the Haar state of O_n^+ is not faithful on $C_u(O_n^+)$. One defines the **reduced $\mathbf{C}^*$-algebra** $C_r(O_n^+)$ **of "cont. functions" on** O_n^+ as the image of the GNS representation of $C_u(O_n^+)$ w.r.t. h.
$\Rightarrow$ By construction h is faithful on $C_r(O_n^+)$.

The $*$-Hopf algebra $\mathrm{Pol}(O_n^+)$ of "polynomials" on O_n^+

$\mathrm{Pol}(O_n^+)$ is the $*$-subalgebra of $C_u(O_n^+)$ or $C_r(O_n^+)$ generated by x_{jk}, $1 \leq j, k \leq n$.
O_n^+ is of Kac type, i.e. the Haar state h is a trace and $S^2 = \mathrm{id}$.

From conv. semigroups to transl.inv. Markov semigroups

Theorem

Let $(\varphi_t)_{t\geq 0}$ be a continuous convolution semigroup of states on $\mathrm{Pol}(\mathbb{G})$, i.e.

$$\forall s,t\geq 0, \quad \varphi_s\star\varphi_t=\varphi_{s+t},$$
$$\forall a\in\mathrm{Pol}(\mathbb{G}), \quad \lim_{t\searrow 0}\varphi_t(a)=\varphi_0(a)=\varepsilon(a).$$

Then the semigroup $(T_t)_{t\geq 0}$,

$$T_t=(\mathrm{id}\otimes\varphi_t)\circ\Delta:\mathrm{Pol}(\mathbb{G})\to\mathrm{Pol}(\mathbb{G})$$

extends continuously to $C_u(\mathbb{G})$ and $C_r(\mathbb{G})$.
The T_t are **translation invariant** in the sense that

$$\Delta\circ T_t=(\mathrm{id}\otimes T_t)\circ\Delta.$$

Theorem

Let $\mathbb{G} = (\mathsf{A}, \Delta)$ be a compact quantum group and $(T_t)_{t\geq 0}$ a Markov semigroup on $\mathsf{A} = C(\mathbb{G})$.
Then $(T_t|_{\mathrm{Pol}(\mathbb{G})})_{t\geq 0}$ is of the form

$$T_t|_{\mathrm{Pol}(\mathbb{G})} = (\mathrm{id} \otimes \varphi_t) \circ \Delta$$

if and only if T_t is translation invariant for all $t \geq 0$.

Corollary

One-to-one correspondence between translation invariant Markov semigroups on $C_r(\mathbb{G})$ (or $C_u(\mathbb{G})$) and **Lévy processes** (in the sense of Schürmann) on $\mathrm{Pol}(\mathbb{G})$.

$\mathfrak{ad}$-Invariance

Definition

The **adjoint action** of a Hopf algebra $\mathcal{A}$ is defined by $\mathrm{ad}: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$,

$$\mathfrak{ad}(a) = a_{(1)} S(a_{(3)}) \otimes a_{(2)}, \quad a \in \mathcal{A}.$$

where $\Delta^{(2)}(a) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$ (Sweedler notation).

Definition

A linear map $T \in \mathrm{Lin}(\mathcal{A})$ is called $\mathfrak{ad}$-**invariant**, if

$$(\mathrm{id} \otimes T) \circ \mathfrak{ad} = \mathfrak{ad} \circ T.$$

A linear functional $L \in \mathcal{A}'$ is called $\mathfrak{ad}$-**invariant**, if

$$(\mathrm{id} \otimes L) \circ \mathfrak{ad} = L\mathbf{1}_{\mathcal{A}}.$$

$\mathfrak{ad}$-*Invariance*

Remarks

- The counit ε and the Haar state h are $\mathfrak{ad}$-invariant.
- For $L \in \mathcal{A}'$, T_L is $\mathfrak{ad}$-invariant if and only if L is $\mathfrak{ad}$-invariant.
- If $L, L' \in \mathcal{A}'$ are $\mathfrak{ad}$-invariant then $L \star L'$ is $\mathfrak{ad}$-invariant.

Proposition

Let $(\varphi_t)_{t\geq 0}$ be a convolution semigroup of unital functionals on $\mathcal{A}$. The following are equivalent.

- The operators $T_t = (\mathrm{id} \otimes \varphi_t) \circ \Delta$ are $\mathfrak{ad}$-invariant;
- The functionals φ_t are $\mathfrak{ad}$-invariant;
- The functional

$$L = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \varphi_t$$

is $\mathfrak{ad}$-invariant.

$\mathfrak{ad}$-inv. functionals and the subalg. of central functions

Let $\mathcal{A} = \mathrm{Pol}(\mathbb{G})$ now be the $*$-Hopf algebra of "polynomials" on a compact quantum group.

Proposition

A functional L is $\mathfrak{ad}$-invariant if and only if it is of the form $L(u_{jk}^{(s)}) = c_s\delta_{jk}$ with some coefficients $c_s \in \mathbb{C}$.

Denote by

$$\mathcal{A}_0 = \mathrm{Lin}\left\{ \chi_s = \sum_{j=1}^{n_s} u_{jj}^{(s)}; s \in \mathcal{I} \right\}$$

the **algebra of "central polynomial functions" on** $\mathbb{G}$.

Proposition

If $\mathbb{G}$ is of Kac type, then $\mathrm{ad}_h : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathrm{ad}_h(a) = (h \otimes \mathrm{id}) \circ \mathrm{ad}(a) = h\big(a_{(1)} S(a_{(3)})\big) a_{(2)}$$

satisfies

$$\mathrm{ad}_h(a^* a) = (h \otimes \mathrm{id})\Big(\big(\mathrm{ad}(a)\big)^* \mathrm{ad}(a)\Big)$$

for $a \in \mathcal{A}$ and therefore preserves positivity.
Furthermore, we have $\mathrm{ad}_h(\mathcal{A}) = \mathcal{A}_0$, $\mathrm{ad}_h \circ \mathrm{ad}_h = \mathrm{ad}_h$, $\mathrm{ad}_h(1) = 1$, $h \circ \mathrm{ad}_h = h$, i.e. ad_h is a conditional expectation onto $\mathcal{A}_0$.

Now we know what we have to do!

Corollary

If $\mathbb{G}$ is of Kac type, then

$$\mathrm{ad}_h^* : \mathcal{A}_0' \ni \phi \mapsto \phi \circ \mathrm{ad}_h \in \mathcal{A}'$$

defines bijections between states on $\mathcal{A}_0$ and $\mathfrak{ad}$-invariant states on $\mathcal{A}$, and between generating functionals on $\mathcal{A}_0$ and ad-invariant $\mathfrak{ad}$-invariant generating functionals on $\mathcal{A}$.

Conclusion

In order to classify translation invariant $\mathfrak{ad}$-invariant generating functionals on a compact quantum group $\mathbb{G}$ of Kac type, it is sufficient to classify the generating functionals on its algebra $\mathcal{A}_0(\mathbb{G})$ of central polynomial functions.

Example: The free orthogonal quantum group O_n^+

For O_n^+, we have

$$\mathcal{A}_0(O_n^+) \cong \mathrm{Pol}([-n, n]).$$

and $\varepsilon(f) = f(n)$ for $f \in \mathcal{A}_0(O_n^+) \cong \mathrm{Pol}([-n, n])$.
The generating functionals on $\mathcal{A}_0(O_n^+) \cong \mathrm{Pol}([-n, n])$ are of the form

$$Lf = -af'(n) + \int_{-n}^{n} \frac{f(x) - f(n)}{n - x} \mathrm{d}\nu(x)$$

where $b > 0$ is a real number and ν a finite measure on $[-n, n]$.

Remark

The same method works for all compact quantum groups of Kac type (with commutative fusion rules), e.g., for the free permutation quantum group S_n^+ we have an analogous formula.

Thank you!

谢谢

Selected references

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