Noncommutative Boyd interpolation theorems

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- joint work with B. de Pagter, D. Potapov, F. Sukochev and with É. Ricard

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A sublinear operator T is of *Marcinkiewicz weak type* (p,p) if for any $f \in L^p(\mathbb{R}_+)$,

$$d(v;Tf)^{\frac{1}{p}} \leq Cv^{-1}\|f\|_{L^p(\mathbb{R}_+)} \qquad (v>0).$$

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where

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Example: if $(\mathbb{E}_n)_{n\geq 1}$ is a sequence of conditional expectations, then $Tf = \sup_{n\geq 1} |\mathbb{E}_n(f)|$ is of M-weak types $(1,1)$ and (∞,∞).

Theorem

(Marcinkiewicz '39) Let $1 \leq p < q \leq \infty$. If T is of Marcinkiewicz weak types (p,p) and (q,q), then T is bounded on $L^r(\mathbb{R}_+)$, for any $p < r < q$.

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Example: if $(\mathbb{E}_n)_{n\geq 1}$ is a sequence of conditional expectations, then for all $1 < p \leq \infty$

$$\|\sup_{n\geq 1} |\mathbb{E}_n(f)|\|_{L^p} \lesssim_p \|f\|_{L^p}.$$

T is said to be of *weak type* (p, p) if for any f in the Lorentz space $L^{p,1}(\mathbb{R}_+)$,

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For $0 < p < q < \infty$ define Calderón's operator by

$$S_{p,q}f(t) = t^{-\frac{1}{p}} \int_0^t s^{\frac{1}{p}} f(s) \frac{ds}{s} + t^{-\frac{1}{q}} \int_t^\infty s^{\frac{1}{q}} f(s) \frac{ds}{s}$$

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Theorem

(Calderón, '66) T is of weak types (p, p) and (q, q) if and only if

$$\mu_t(Tf) \lesssim_{p,q} \Big(S_{p,q}\mu(f)\Big)(t) \qquad (t > 0).$$

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$$p_E := \lim_{s\to\infty} \frac{\log s}{\log \|D_{1/s}\|}, \quad q_E := \lim_{s\downarrow 0} \frac{\log s}{\log \|D_{1/s}\|}$$

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The last implication '$\Leftarrow$' is Boyd's interpolation theorem.

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Here $\phi_\infty = \chi_{(0,1)}$. Corresponding to these functions we define three linear operators $\Phi_q, \Psi_p, \Theta_{p,q} : S(\mathbb{R}_+) \to \tilde{S}(\mathbb{R}_+ \times \mathbb{R}_+)$ by

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3. *If $p < p_E \leq q_E < q$, then $\Theta_{p,q}$ is bounded from $E(\mathbb{R}_+)$ into $E(\mathbb{R}_+ \times \mathbb{R}_+)$. Conversely, if $\Theta_{p,q}$ is bounded, then $p \leq p_E \leq q_E \leq q$.*

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This lemma implies a novel expression for Boyd's indices:

$$p_E = \sup\left\{p > 0 \ : \ \exists C > 0 \ \forall f \in E \ \|\Psi_p(f)\|_{E(\mathbb{R}_+ \times \mathbb{R}_+)} \leq C\|f\|_{E(\mathbb{R}_+)}\right\}$$

$$q_E = \inf\left\{q > 0 \ : \ \exists C > 0 \ \forall f \in E \ \|\Phi_q(f)\|_{E(\mathbb{R}_+ \times \mathbb{R}_+)} \leq C\|f\|_{E(\mathbb{R}_+)}\right\}.$$

Theorem

Let $0 < p \leq q \leq \infty$. A sublinear operator T is of Marcinkiewicz weak types (p,p) and (q,q), i.e.,

$$\|Tf\|_{L^{r,\infty}(\mathbb{R}_+)} \leq C_r \|f\|_{L^r(\mathbb{R}_+)} \qquad (f \in L^r(\mathbb{R}_+)_+, r = p, q)$$

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if and only if there is some $\alpha > 0$ such that for all $f \in S(\mathbb{R}_+)$,

$$d(\alpha v; Tf) \leq d(v; \Theta_{p,q}(f)) \qquad (v > 0).$$

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$$\begin{aligned} &d(2C_{p,q}v; Tf) \\ &\quad\leq (C_{p,q}v)^{-q} C_q^q \|f\chi_{\{f\leq v\}}\|^q_{L^q(\mathbb{R}_+)} + (C_{p,q}v)^{-p} C_p^p \|f\chi_{\{f>v\}}\|^p_{L^p(\mathbb{R}_+)} \\ &\quad\leq v^{-q}\|f\chi_{\{f\leq v\}}\|^q_{L^q(\mathbb{R}_+)} + v^{-p}\|f\chi_{\{f>v\}}\|^p_{L^p(\mathbb{R}_+)} \\ &\quad= d(v; \Theta_{p,q}f). \end{aligned}$$

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This characterization implies Boyd's theorem for Marcinkiewicz weak type operators.

Corollary

If T is of Marcinkiewicz weak types (p,p) and (q,q) and either $p < p_E \leq q_E < q < \infty$ or $p < p_E$ and $q = \infty$, then

$$\|Tf\|_E \leq 2\|\Theta_{p,q}\|_{E\to E} \max\{C_p, C_q\}\|f\|_E.$$

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If $0 < p < r < q < \infty$, then

$$\|\Theta_{p,q}\|_{L^r\to L^r} = \left(\frac{p}{r-p} + \frac{q}{q-r}\right)^{\frac{1}{r}}.$$

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We say that x is τ-*measurable* if $d(v;x) < \infty$ for some $v > 0$.

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$$E(\mathcal{M},\tau) := \{x \in S(\tau) : \ \|\mu(x)\|_E < \infty\}.$$

Theorem

(Kalton & Sukochev '08) $E(\mathcal{M})$ defines a Banach space under the norm $\|x\|_{E(\mathcal{M})} := \|\mu(x)\|_E$.

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Earlier results: Xu ('91), Dodds, Dodds & de Pagter ('91).

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Suppose that $0 < p < q \leq \infty$ *and let* $T : L^p(\mathcal{M})_+ + L^q(\mathcal{M})_+ \to S(\tau)_+$ *be a sublinear map such that*

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If $p < p_E \leq q_E < q < \infty$ *or* $p < p_E$ *and* $q = \infty$, *then*

$$\|Tx\|_{E(\mathcal{M})} \leq 2\|\Theta_{p,q}\| \max\{C_p, C_q\}\ \|x\|_{E(\mathcal{M})} \qquad (x \in E(\mathcal{M})_+).$$

Proof.

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$$\begin{aligned} & d(2C_{p,q}v; Tx) \\ & \quad \leq (C_{p,q}v)^{-q} C_q^q \|xe_v\|_{L^q(\mathcal{M})}^q + (C_{p,q}v)^{-p} C_p^p \|xe_v^{\perp}\|_{L^p(\mathcal{M})}^p \\ & \quad \leq v^{-q} \int_{\{\mu(x) \leq v\}} \mu_t(x)^q dt + v^{-p} \int_{\{\mu(x) > v\}} \mu_t(x)^p dt \\ & \quad = d(v; \Theta_{p,q}\mu(x)). \end{aligned}$$

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This result can be viewed as a Boyd interpolation theorem for operators on $L^r(\mathcal{M}; l^1)$. By duality one can interpolate noncommutative maximal inequalities.

Corollary

Let $(\mathcal{E}_k)_{k\geq 1}$ be an increasing sequence of conditional expectations in $\mathcal{M}$. If $1 < p_E \leq q_E < \infty$, then for any sequence $(x_k)_{k\geq 1}$ in $E(\mathcal{M})_+$,

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This follows by interpolation from Junge's ('02) result for L^p-spaces.

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Complements the 'lower' Khintchine inequality due to Le Merdy & Sukochev ('08).

A symmetric space E on $\mathbb{R}_+$ is called *q-concave* (for $q < \infty$) if

$$\Big(\sum_i \|f_i\|_E^q\Big)^{\frac{1}{q}} \leq C\Big\|\Big(\sum_i |f_i|^q\Big)^{\frac{1}{q}}\Big\|_E.$$

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Extends earlier Burkholder-Rosenthal inequalities of Junge & Xu ('03), Jiao ('10) and D., de Pagter, Potapov & Sukochev ('11).