Noncommutative integral inequalities for convex functions of maximal functions and applications

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1. Cuculescu (1971) proved the Doob inequality of weak type $(1,1)$ for noncommutative martingales.
2. Junge (2002) established the Doob inequality of type (p,p) for noncommutative martingales.
3. Junge/Xu (2007) proved the maximal ergodic inequalities on noncommutative L_p-spaces.

Aim

Our goal is to prove the noncommutative analogue of the Doob inequality for convex functions of maximal functions.

For this, we establish a Marcinkiewicz type interpolation theorem for convex functions of maximal functions in the noncommutative setting.

Noncommutative Orlicz spaces

Let $\mathcal{N}$ be a semifinite von Neumann algebra acting on a Hilbert space $\mathbb{H}$ with a normal semifinite faithful trace ν. Let $L_0(\mathcal{N})$ denote the topological $*$-algebra of measurable operators with respect to $(\mathcal{N},\nu)$. The topology of $L_0(\mathcal{N})$ is determined by the convergence in measure.
For $x \in L_0(\mathcal{N})$ we define

$$\lambda_s(x) = \tau(e_s^{\perp}(|x|)) \ (s > 0)$$

and

$$\mu_t(x) = \inf\{s > 0 : \ \lambda_s(x) \le t\} \ (t > 0),$$

where $e_s^{\perp}(|x|) = e_{(s,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (s,∞). The function $s \mapsto \lambda_s(x)$ is called the *distribution function* of x and $\mu_t(x)$ is the *generalized singular number* of x.

Noncommutative Orlicz spaces

Let Φ be an Orlicz function on $[0,\infty)$, i.e., a continuous increasing and convex function satisfying $\Phi(0)=0$ and $\lim_{t\to\infty}\Phi(t)=\infty$. Recall that Φ is said to satisfy the $\triangle_2$-condition if there is a constant C such that $\Phi(2t)\le C\Phi(t)$ for all $t>0$. In this case, we write $\Phi\in\Delta_2$.

Given an Orlicz function Φ, let

$$M(t,\Phi)=\sup_{s>0}\frac{\Phi(ts)}{\Phi(s)},\quad t>0.$$

Define

$$p_\Phi=\lim_{t\searrow 0}\frac{\log M(t,\Phi)}{\log t},\quad q_\Phi=\lim_{t\nearrow\infty}\frac{\log M(t,\Phi)}{\log t}.$$

Noncommutative Orlicz spaces

The following characterizations of p_Φ and q_Φ hold

$$p_\Phi = \sup\left\{p>0:\ \int_0^t s^{-p}\Phi(s)\frac{ds}{s} = O(t^{-p}\Phi(t)),\ \forall t>0\right\};$$

$$q_\Phi = \inf\left\{q>0:\ \int_t^\infty s^{-q}\Phi(s)\frac{ds}{s} = O(t^{-q}\Phi(t)),\ \forall t>0\right\}.$$

For an Orlicz function Φ, the noncommutative Orlicz space $L_\Phi(\mathcal{N})$ is defined as the space of all measurable operators x with respect to $(\mathcal{N},\nu)$ such that

$$\nu\left(\Phi\left(\frac{|x|}{c}\right)\right) < \infty$$

for some $c>0$.

Noncommutative Orlicz spaces

The space $L_\Phi(\mathcal{N})$, equipped with the norm

$$\|x\|_\Phi = \inf\Big\{c > 0 :\ \nu\big(\Phi(|x|/c)\big) < 1\Big\},$$

is a Banach space. If $\Phi(t) = t^p$ with $1 \leq p < \infty$ then $L_\Phi(\mathcal{N}) = L_p(\mathcal{N})$. Note that if $\Phi \in \triangle_2$, then for $x \in L_0(\mathcal{N})$, $\nu(\Phi(x)) < \infty$ if and only if $x \in L_\Phi(\mathcal{N})$.

$L_p(\mathcal{M};\ell^\infty)$ spaces

Given $1 \le p < \infty$, recall that $L_p(\mathcal{M};\ell^\infty)$ is defined as the space of all sequences $(x_n)_{n\ge1}$ in $L_p(\mathcal{M})$ for which there exist $a, b \in L_{2p}(\mathcal{M})$ and a bounded sequence $(y_n)_{n\ge1}$ in $\mathcal{M}$ such that $x_n = ay_nb$ for all $n \ge 1$. For such a sequence, set

$$\|(x_n)_{n\ge1}\|_{L_p(\mathcal{M},\ell^\infty)} := \inf\Big\{\|a\|_{2p} \sup_n \|y_n\|_\infty \|b\|_{2p}\Big\}, \qquad (2.1)$$

where the infimum runs over all possible factorizations of $(x_n)_{n\ge1}$ as above. This is a norm and $L_p(\mathcal{M};\ell^\infty)$ is a Banach space.

$L_p(\mathcal{M};\ell^\infty)$ spaces

These spaces were first introduced by Pisier (1998) in the case when $\mathcal{M}$ is hyperfinite and by Junge (2002) in the general case. It is easy to check that

$$\|(x_n)_{n\geq 1}\|_{L_p(\mathcal{M},\ell^\infty)} = \inf\left\{\frac{1}{2}\left(\|a\|_{2p}^2 + \|b\|_{2p}^2\right)\sup_n \|y_n\|_\infty\right\}, \tag{2.2}$$

the infimum taken over the same parameters as above.
We usually write

$$\left\|\sup_n{}^+ x_n\right\|_p = \|(x_n)_{n\geq 1}\|_{L_p(\mathcal{M},\ell^\infty)}.$$

$L_p(\mathcal{M};\ell^\infty)$ spaces

Definition

Let Φ be an Orlicz function. Let (x_n) be a sequence in $L_\Phi(\mathcal{M})$. We define $\tau[\Phi(\sup_n{}^+ x_n)]$ by

$$\tau\left[\Phi\left(\sup_n{}^+ x_n\right)\right] := \inf\left\{\frac{1}{2}\left(\tau\left[\Phi\left(|a|^2\right)\right] + \tau\left[\Phi\left(|b|^2\right)\right]\right)\sup_n \|y_n\|_\infty\right\} \tag{2.3}$$

where the infimum is taken over all decompositions $x_n = ay_nb$ for $a, b \in L_0(\mathcal{M})$ and $(y_n) \subset L_\infty(\mathcal{M})$ with $|a|^2, |b|^2 \in L_\Phi(\mathcal{M})$, and $\|y_n\|_\infty \leq 1$ for all n.

$L_p(\mathcal{M};\ell^\infty)$ spaces

Proposition

Let Φ *be an Orlicz function satisfying the* $\triangle_2$*-condition.*

(1) *If* $x=(x_n)$ *is a positive sequence in* $L_\Phi(\mathcal{M})$, *then*

$$\tau\Big[\Phi\big(\sup_n{}^+ x_n\big)\Big] \approx \inf\Big\{\tau\big[\Phi(a)\big] : \ a\in L_\Phi^+(\mathcal{M}) \text{ such that } x_n\le a, \forall n \Big\}$$

(2) *For any two sequences* $x=(x_n), y=(y_n)$ *in* $L_\Phi(\mathcal{M})$ *one has*

$$\tau\Big[\Phi\big(\sup_n{}^+ (x_n+y_n)\big)\Big] \lesssim \tau\Big[\Phi\big(\sup_n{}^+ x_n\big)\Big] + \tau\Big[\Phi\big(\sup_n{}^+ y_n\big)\Big].$$

$L_p(\mathcal{M};\ell^\infty)$ spaces

Remark

For a sequences $x=(x_n)$ in $L_\Phi(\mathcal{M})$, set

$$\Big\|{\sup_n}^+ x_n\Big\|_\Phi := \inf\left\{\lambda>0:\ \tau\Big[\Phi\Big({\sup_n}^+\frac{x_n}{\lambda}\Big)\Big]\le 1\right\}.$$

One can check that $\|{\sup_n}^+ x_n\|_\Phi$ is a norm in $x=(x_n)$. Define

$$L_\Phi(\mathcal{M};\ell^\infty) := \left\{(x_n)\subset L_\Phi(\mathcal{M}):\ \tau\Big[\Phi\Big({\sup_n}^+\frac{x_n}{\lambda}\Big)\Big]<\infty\ \exists\lambda>0\right\},$$

equipped with $\|(x_n)\|_{L_\Phi(\mathcal{M};\ell^\infty)}=\|{\sup_n}^+ x_n\|_\Phi$. Then $L_\Phi(\mathcal{M};\ell^\infty)$ is a Banach space. For $1\le p<\infty$, if $\Phi(t)=t^p$ then $L_\Phi(\mathcal{M};\ell^\infty)=L_p(\mathcal{M};\ell^\infty)$.

interpolation

Definition

Let $1 \le p_0 < p_1 \le \infty$. *Let* $S = (S_n)_{n\ge 1}$ *be a sequence of maps from* $L^+_{p_0}(\mathcal{M}) + L^+_{p_1}(\mathcal{M}) \mapsto L^+_0(\mathcal{M})$.

(1) S *is said to be subadditive, if for any* $n \ge 1$,

$$S_n(x+y) \le S_n(x) + S_n(y), \quad \forall x, y \in L^+_{p_0}(\mathcal{M}) + L^+_{p_1}(\mathcal{M}).$$

(2) S *is said to be of weak type* (p,p) ($p_0 \le p < p_1$) *if there is a positive constant* C *such that for any* $x \in L^+_p(\mathcal{M})$ *and any* $\lambda > 0$ *there exists a projection* $e \in \mathcal{M}$ *such that*

$$\tau(e^\perp) \le \left(\frac{C\|x\|_p}{\lambda}\right)^p \quad \text{and} \quad eS_n(x)e \le \lambda,\ \forall n \ge 1.$$

interpolation

(3) S is said to be of type (p,p) $(p_0 \le p \le p_1)$ if there is a positive constant C such that for any $x \in L_p^+(\mathcal{M})$ there exists $a \in L_p^+(\mathcal{M})$ satisfying

$$\|a\|_p \le C\|x\|_p \quad \text{and} \quad S_n(x) \le a, \ \forall n \ge 1.$$

In other words, S is of type (p,p) if and only if $\|S(x)\|_{L_p(\mathcal{M};\ell^\infty)} \le C\|x\|_p$ for all $x \in L_p^+(\mathcal{M})$.

interpolation

Theorem (B-Chen-Osękowski)

Let $S=(S_n)_{n\geq 0}$ be a sequence of maps from $L_1^+(\mathcal{M})+L_\infty^+(\mathcal{M})\mapsto L_0^+(\mathcal{M})$. Let $1\leq p<\infty$. Assume that S is subadditive. If S is simultaneously of weak type (p,p) with constant C_p and of type (∞,∞) with constant C_∞, then for an Orlicz function Φ with $p<p_\Phi\leq q_\Phi<\infty$, there exists a positive constant C depending only on C_p, C_∞, p_Φ and q_Φ, such that

$$\tau\left[\Phi\left(\sup_n{}^+ S_n(x)\right)\right]\leq C\tau\left[\Phi(x)\right], \tag{3.1}$$

for all $x\in L_\Phi^+(\mathcal{M})$.

Doob inequality

Let $\mathcal{M}$ be a finite von Neumann algebra with a normalized normal faithful trace τ. Let $(\mathcal{M}_n)_{n\geq 0}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that $\cup_{n\geq 0}\mathcal{M}_n$ generates $\mathcal{M}$ (in the w^*-topology).

$(\mathcal{M}_n)_{n\geq 0}$ is called a filtration of $\mathcal{M}$. The restriction of τ to $\mathcal{M}_n$ is still denoted by τ. Let $\mathcal{E}_n = \mathcal{E}(\cdot|\mathcal{M}_n)$ be the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_n$. Then $\mathcal{E}_n$ is a norm 1 projection of $L_\Phi(\mathcal{M})$ onto $L_\Phi(\mathcal{M}_n)$ and $\mathcal{E}_n(x)\geq 0$ whenever $x\geq 0$.

A noncommutative L_Φ-martingale with respect to $(\mathcal{M}_n)_{n\geq 0}$ is a sequence $x=(x_n)_{n\geq 0}$ such that $x_n\in L_\Phi(\mathcal{M}_n)$ and

$$\mathcal{E}_n(x_{n+1}) = x_n$$

for any $n\geq 0$.

Let $\|x\|_\Phi = \sup_{n\geq 0}\|x_n\|_\Phi$. If $\|x\|_\Phi<\infty$, then x is said to be a bounded L_Φ-martingale.

Doob inequality

Theorem (B-Chen-Osękowski)

Let $\mathcal{M}$ be a finite von Neumann algebra with a normalized normal faithful trace τ, equipped with a filtration $(\mathcal{M}_n)_{n\geq 0}$ of von Neumann subalgebras of $\mathcal{M}$. Let Φ be an Orlicz function and $x=(x_n)$ be a noncommutative L_Φ-martingale with respect to $(\mathcal{M}_n)$. If $1<p_\Phi\leq q_\Phi<\infty$, then

$$\tau\left[\Phi\left(\sup_n{}^+ x_n\right)\right]\approx\tau\left[\Phi(|x|)\right]. \tag{4.1}$$